Algorithm Design Techniques

- Induction
- Divide and Conquer
- Dynamic Programming
- **Greedy**
- Reduction

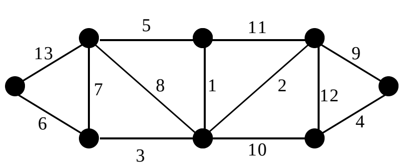
Greedy Paradigm
Get what you can NOW!

But sometimes it's better to look around!

But sometimes it isn't …

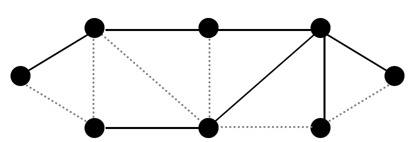
Minimum Spanning Tree

- G=(V,E) is a connected, weighted graph with n vertices and m edges.



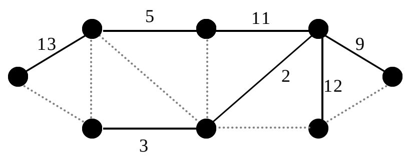
Minimum Spanning Tree-cont.

- A spanning tree of G is a connected, acyclic subgraph with vertex set V.



Minimum Spanning Tree-cont.

- The weight of spanning tree of G is the sum of the weights of its edges.



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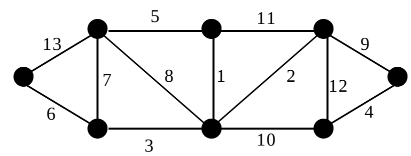
Minimum Spanning Tree-cont.

- A minimum spanning tree of G is one with smallest possible weight.
- Find an MST of the following graph:



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Kruskal's (Greedy) Algorithm

Let $e_1, e_2, \dots, e_m$ be the edges of G sorted by increasing weight.

F=V (F is a forest of isolated vertices)

For i=1 to m

  If F+{e_i} is acyclic then F=F+{e_i}.

Return(F)

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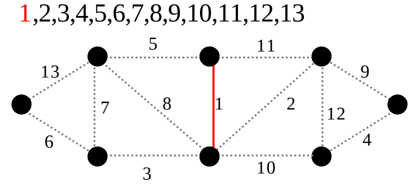
Kruskal's Algorithm

- Order the edge weights. (In this graph the weights are unique.)
- 1,2,3,4,5,6,7,8,9,10,11,12,13



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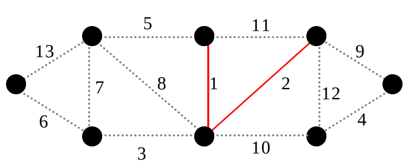
Kruskal's Algorithm-cont.

- 1,2,3,4,5,6,7,8,9,10,11,12,13



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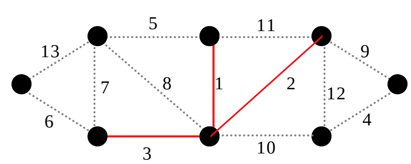
Kruskal's Algorithm-cont.

- 1,2,3,4,5,6,7,8,9,10,11,12,13



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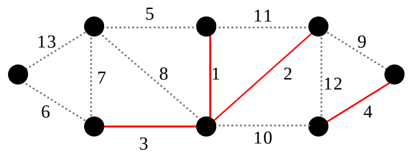
Kruskal's Algorithm-cont.

- 1,2,3,4 ,5,6,7,8,9,10,11,12,13



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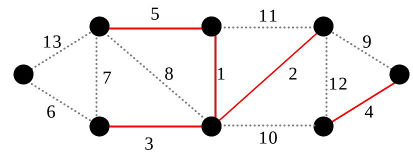
Kruskal's Algorithm-cont.

- 1,2,3,4,5,6,7,8,9,10,11,12,13



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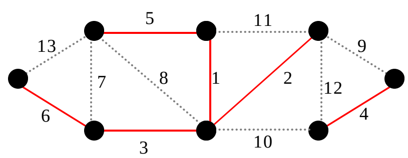
Kruskal's Algorithm-cont.

- 1,2,3,4,5,6 ,7,8,9,10,11,12,13



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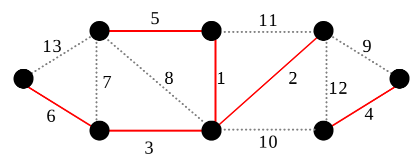
Kruskal's Algorithm-cont.

- 1,2,3,4,5,6,**7**,8,9,10,11,12,13
- Can we add the edge with weight 7?



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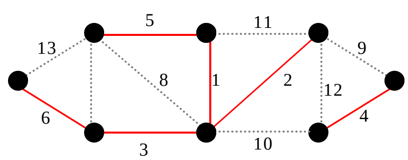
Kruskal's Algorithm-cont.

- 1,2,3,4,5,6,7,8,9,10,11,12,13
- Can we add the edge with weight 8?



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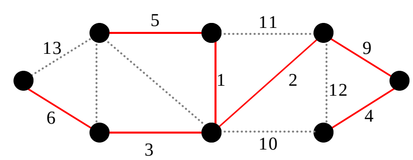
Kruskal's Algorithm-cont.

- 1,2,3,4,5,6,7,8,9,10,11,12,13



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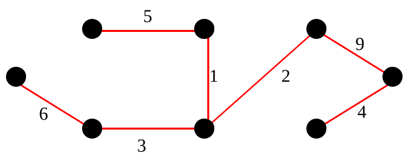
Kruskal’s Algorithm-cont.

MST of G with cost ________



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Kruskal’s Algorithm

- Does it work in general?
- Prove it.

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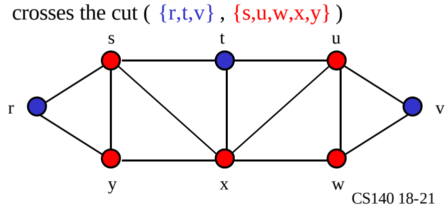
Cut

A **cut** is a partition of the vertices of G into two sets (S,S’). An edge e crosses the cut if it has an endpoint in each set of the cut. For example (t,x) crosses the cut ({r,t,v} , {s,u,w,x,y})



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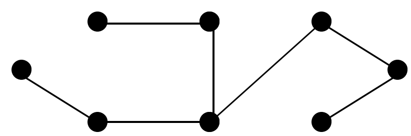
Tree Facts

- A tree on n nodes has n -1 edges.



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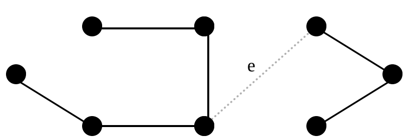
Tree Facts

- If e is an edge of T then T-{e} is a forest consisting of two trees.



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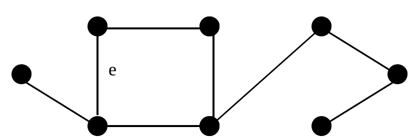
Tree Facts

- If e is an edge of G but not of T then T+{e} contains exactly one cycle.



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Tree Facts

1. A tree on n nodes has n -1 edges.
2. If e is an edge of T then T-{e} is a forest consisting of two trees.
3. If e is an edge of G but not of T then T+{e} contains exactly one cycle.

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Kruskal's Algorithm Proof of Correctness

Claim:

At each stage of the algorithm F is a subgraph of some MST of G.

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Kruskal's Algorithm

Let $e_1, e_2, \dots, e_m$ be the edges of G sorted by increasing weight.

F=V (F is a forest of isolated vertices)

Claim is true here

For i=1 to m

If F+{e_i} is acyclic then F=F+{e_i }.

Return(F)

CS140 18-27

Loop Invariant

Let $e_1, e_2, \dots, e_m$ be the edges of G sorted by increasing weight.

F=V (F is a forest of isolated vertices) Claim is true here

If Claim is true here

For i=1 to m

If F+{e_i} is acyclic then F=F+{e_i}.

Then Claim is true here

Return(F)

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Kruskal's Algorithm Proof of Correctness

Loop Invariant:

F is a subgraph of some MST of G .

Proof:

Consider the k[th] execution of the loop. Let T be a MST of G containing F. What can happen during the loop?

1. e_k is not added to F
2. e_k is added to F

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Kruskal's Algorithm Proof of Correctness

Loop Invariant:

F is a subgraph of some MST of G.

Proof:

1. e_k is not added to F

   In this case F does not change so the claim holds when execution of loop concludes

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Kruskal's Algorithm Proof of Correctness

Loop Invariant:

F is a subgraph of some MST of G.

Proof:

2. e_k is added to F
   We know that T is a MST of G and T contains F.
   Need to show there is a MST of G that contains F+{ e_k }
   If e_k is an edge of T we are done. So assume not.

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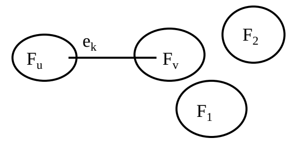
What do we know?

Assume e_k=(u,v). The vertices u and v are in separate connected components. Let S be the vertices of F_u.



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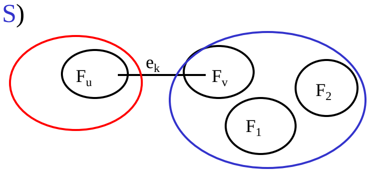
What do we know?

e_k is a minimum weight edge spanning (S,V-S)



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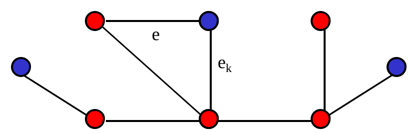
Using our tree facts

- The graph T+{e_k} contains exactly one cycle.
- This cycle contains e_k and at least one additional edge e that spans (S,V-S).
- T+{e_k}-{e} is an MST of G.



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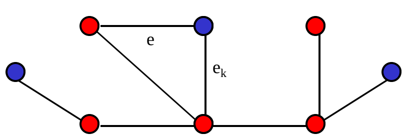
Moreover

- T+{e_k}-{e} is an MST of G **that contains the edges of F+{e_k}.**



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