MST

- If G is connected graph with distinct edge weights then G has a unique MST.
- Proof: Follows from the cut theorem.

CS140 20-1

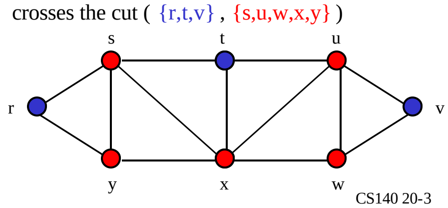
Cut Theorem

Cut Theorem: Let (S,S’) be a cut of G. If e is a minimum weight edge spanning the cut, then e is in some MST of G. Further, if the minimum weight edge spanning the cut is unique, then e is in every MST of G.

CS140 20-2

Cut

A **cut** is a partition of the vertices of G into two sets (S,S’). An edge e crosses the cut if it has an endpoint in each set of the cut. For example (t,x) crosses the cut ({r,t,v} , {s,u,w,x,y})



CS140 20-3

Tree Facts

1. A tree on n nodes has n-1 edges.
2. If e is an edge of T then T-{e} is a forest consisting of two trees.
3. If e is an edge of G but not of T then T+{e} contains exactly one cycle.

CS140 20-4

Proof of Cut Theorem

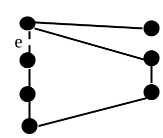
Cut Theorem: Let (S,S’) be a cut of G. If e is a minimum weight edge spanning the cut, then e is in some MST of G. Further, if the minimum weight edge spanning the cut is unique, then e is in every MST of G.

Proof: Let T be a minimum spanning tree of G. Suppose that e is not an edge of T.

CS140 20-5

Proof of Cut Theorem

- Then T+{e} contains exactly one cycle.

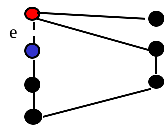


T+{e}

CS140 20-6

Proof of Cut Theorem cont.

- Since e crosses the cut (S,S')

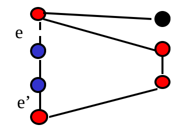


T+{e}

CS140 20-7

Proof of Cut Theorem cont.

- Since e crosses the cut (S,S'), another edge in the cycle must cross the cut too.



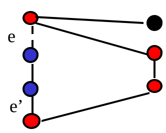
T+{e}

CS140 20-8

Proof of Cut Theorem cont.

- Note that w(e) $\leq$ w(e') since e is a minimum weight edge crossing the cut (S,S')

(We'll use this in a minute.)

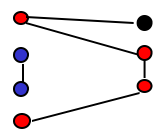


T+{e}

CS140 20-9

Proof of Cut Theorem cont.

- T-{e'} is a forest consisting of two trees.

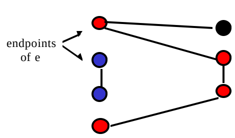


T-{e'}

CS140 20-10

Proof of Cut Theorem cont.

- The endpoints of e are in different connected components of T-{e'}

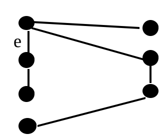


T-{e'}

CS140 20-11

Proof of Cut Theorem cont.

- T'=T-{e'}+{e} is a spanning tree of G

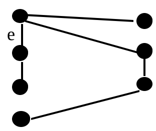


T'=T-{e'}+{e}

CS140 20-12

Proof of Cut Theorem cont.

- w(T’) = w(T-{e’}+{e}) = w(T)-w(e’)+w(e)

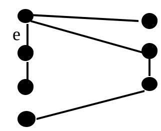


T’=T-{e’}+{e}

CS140 20-13

Proof of Cut Theorem cont.

- w(T’) = w(T-{e’}+{e}) = w(T)-w(e’)+w(e) $\leq$ w(T) because w(e)$\leq$w(e’)

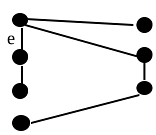


T’=T-{e’}+{e}

CS140 20-14

Proof of Cut Theorem cont.

- T-{e’}+{e} is a MST of G that includes e



T’=T-{e’}+{e}

CS140 20-15

Proof of Cut Theorem cont.

- If the minimum weight edge spanning the cut is unique then
- w(T’) = w(T-{e’}+{e}) = w(T)-w(e’)+w(e) < w(T) because w(e)<w(e’)
- This contradicts the fact that T is a MST of G
- Our only assumption was that e $\notin$ T
- So e ε T ➔ e is in every MST of G

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