Outline

- **A lower bound for comparison-based sorting**
- Linear time sorting

Run Time Bounds

Upper bounds on the *Worst Case* running times of sorting algorithms:

- Bubble-sort: $O(n^2)$
- Modified Bubble-sort: $O(n^2)$
- Insertion-sort: $O(n^2)$
- Merge-sort $O(n \log(n))$
- Heap-sort $O(n \log(n))$
- Quick-sort $O(n^2)$

Run Time Bounds cont.

Lower bounds on the *Worst Case* running times of sorting algorithms:

- Bubble-sort: $\Omega(n^2)$
- Modified Bubble-sort: $\Omega(n^2)$
- Insertion-sort: $\Omega(n^2)$
- Merge-sort: $\Omega(n \log(n))$
- Heap-sort: $\Omega(n \log(n))$
- Quick-sort: $\Omega(n^2)$

Comparison-based sorting

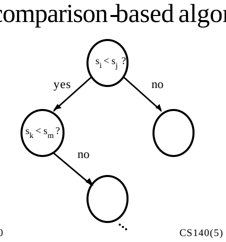
A comparison-based sorting algorithm is one that doesn't need to read the input, provided it is given the size of the input and a comparison oracle.

Lower Bound for Sorting

Theorem: Any comparison-based sorting algorithms has a worst-case running time that is $\Omega(n \log(n))$.

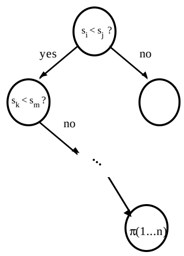
Proof of Theorem

A decision tree describes the queries of a comparison-based algorithm on input size n.



A root to leaf path represents the sequence of queries for a particular input.

Proof of Theorem cont.



Each leaf corresponds to the permutation that sorts the input.

Proof cont.

- There must be at least n! leaves.
- A binary tree with n! leaves has a path with length at least log(n!).
- By Stirling's approximation

$$\log(n!)=\Omega(n \log(n)).$$

Outline

- A lower bound for comparison-based sorting
- **Linear time sorting**

Sorting in O(n)

Can we do it?

Yes – but only if we can make some assumptions about the input.

Some examples:

- Counting-sort
- Radix-sort
- Bucket-sort

Counting-sort(S)

- Assumption: The input integers are in the range [0..B], where B is some constant.

- For i=0 to B: Count(i)=0
- For i=1 to n: Count(S(i))++
- For i=1 to B: For j=1 to count(i): Write i

Counting-sort

- Is it correct?
  - Yes
- Is it fast?
  - O(n+B) but B is constant so O(n)

Radix-sort

- Assumption: The input are decimal integers with exactly D digits, where D is some constant.

- For i=1 to D
  - Use counting-sort to sort S based on i^{th} digit using current order to break ties. (Least significant digit is digit 1.)

Radix-sort(S)

329	720	720	329
457	355	329	355
657	436	436	436
839	457	839	457
436	657	355	657
720	329	457	720
355	839	657	839

Radix-sort

- Is it fast?
  - O(Dn) but D is constant so O(n)

- Is it correct?
  - Loop invariant: The red numbers are sorted.

Radix-sort(S)

329	720	720	329
457	355	329	355
657	436	436	436
839	457	839	457
436	657	355	657
720	329	457	720
355	839	657	839

Bucket-sort(S)

- Assumption: The input are uniformly distributed in the range [0..1).

- Initialize buckets B[0..n-1] to nil
- For i=1 to n
  - Insert S[i] into the sorted list at B[$\lfloor n \cdot S[i] \rfloor$]
  - Output the concatenated buckets

Bucket-sort

- Is it correct?
  - Yes

- Is it fast?
  - The expected running time is O(n).

Average-case analysis

What does average-case mean?

- **Deterministic algorithm with a known input distribution**
- Randomized algorithm on any (i.e. worst-case) input

A brief tour of (discrete) probability theory…

- Sample space and elementary events
- Discrete probability distributions
- Discrete random variables
- Expectation

The experiment

- A fair coin is flipped
- Sample space: {Head, Tail}

The experiment

- Two fair coins are flipped
- Sample space: {HH, HT, TH, TT}

Discrete Probability Distribution

Assigns a real number to outcomes:

- Experiment 1: P(H)=P(T)=1/2
- Experiment 2:

  P(HH)=P(HT)=P(TH)=P(TT)=1/4

Discrete Probability Distribution

- $P(A) \geq 0$
- $P(S) = 1$
- $P(A \cup B)=P(A)+P(B)$ when A and B are disjoint events.

Discrete Random Variable X

- Sample space is a finite set of real numbers.
- X is the outcome of the experiment
- Probability distribution: $P(X=x)$
- Expected value of X:

  $E[X] = \sum_{x \varepsilon S} x\, P(X=x)$

  $E[X^2]= \sum_{x \varepsilon S} x^2\, P(X=x)$

  $Var(X) = E[(X-E[X])^2]$

Example

- A unfair coin is tossed n times:

  $P(H)=p$, $P(T)=1-p=q$
- X is the number of heads
- $P(X=k) = C(n,k)p^k q^{n-k}$

  (Binomial distribution)
- $E[X] = np$
- $E[X^2] = npq + n^2p^2$

Bucket-sort(S)

- Let T be the number of comparisons made by the algorithm on a random input of size n.
- Claim: $E[T] = O(n)$

Analysis

- Let X_i be the number of input that belong in bucket i. X_i is binomially distributed with $p=1/n$.
- $T \le c \sum_{i=0..n-1} X_i^2$
- $E[T] \le c \sum_{i=0..n-1} E[X_i^2]$

  $= c \sum_{i=0..n-1}(npq+n^2p^2) = O(n)$.