

Algorithm Design Techniques

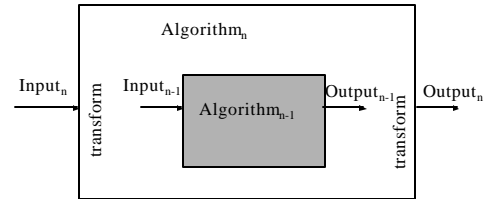
- Induction
- **Divide and Conquer**
- Dynamic Programming
- Greedy
- Reduction

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Induction: Algorithm_n

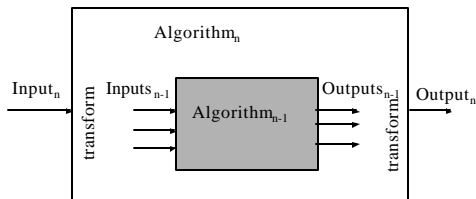


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Induction: Algorithm_n

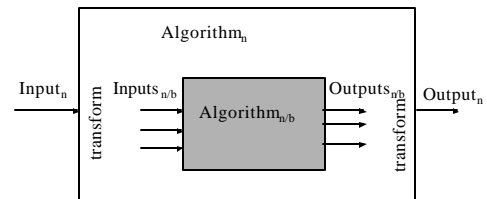


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Divide and Conquer: Algorithm_n



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Sorting

- Insertion Sort: $T(n)=T(n-1)+cn=O(n^2)$
 - Sort first $n-1$ elements
 - Insert element n into sorted list
- Merge Sort: $T(n)=2T(n/b)+cn=O(n \lg(n))$
 - Sort first $n/2$ elements
 - Sort last $n/2$ elements
 - Merge sorted lists into one sorted list

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Matrix Multiplication

- Input: Two $n \times n$ matrices A, B
- Output: The product AB

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Reminder

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Example

- Input:

-1	3
2	4

2	-3
2	1
- Output:

4	6
12	-2

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Algorithm

```

For i = 1 to n
  For j = 1 to n
    C[i,j]=0
    For k = 1 to n
      C[i,j] = C[i,j] + A[i,k]B[k,j]
  
```

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Operation Count

```

For i = 1 to n
  For j = 1 to n
    C[i,j]=0
    For k = 1 to n
      C[i,j] = C[i,j] + A[i,k]B[k,j]
  
```

Requires n^3 multiplications and $n^2(n-1)$ additions.
 (Note the input size is $2n^2$.)

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Divide and Conquer: $n=2^m$

- Input:

A_{11}	A_{12}
A_{21}	A_{22}

B_{11}	B_{12}
B_{21}	B_{22}
- Output:

C_{11}	C_{12}
C_{21}	C_{22}

 $C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j}$

$T(2n^2) = 8T(2(n/2)^2) + cn^2 = O(n^3)$

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Can we do better?

- $T(2n^2) = aT(2(n/2)^2) + cn^2 = o(n^3)$ if $a < 8$
- Strassen's algorithm:

$a=7$ and $T(n) = O(n^{\log_2 7})$

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Strassen's Algorithm Base Case: 2x2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} ae+bf & ag+bh \\ ce+df & cg+dh \end{bmatrix}$$

Is there hope?

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Rephrase

Call this matrix A Call this vector v

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} ae+bf \\ ce+df \\ ag+bh \\ cg+dh \end{bmatrix}$$

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A little magic ...

We can compute Av with 7 scalar multiplications!

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B and C

$$\begin{matrix} B & C & B+C \\ \begin{bmatrix} b & b & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & c \\ 0 & 0 & c & c \end{bmatrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{matrix}$$

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B, C and D

$$\begin{matrix} B+C & D & B+C+D \\ \begin{bmatrix} b & b & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & c & c \\ 0 & 0 & c & c \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ c-b & 0 & 0 & c-b \\ b-c & 0 & 0 & b-c \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{matrix}$$

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B, C, D and E

$$\begin{matrix} B+C+D & E & B+C+D+E \\ \begin{bmatrix} b & b & 0 & 0 \\ c & b & 0 & c-b \\ b-c & 0 & c & b \\ 0 & 0 & c & c \end{bmatrix} & \begin{bmatrix} a-b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c-b & 0 & a-c & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{matrix}$$

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B, C, D, E and F

$$\begin{array}{c}
 \mathbf{B+C+D+E} \quad \mathbf{F} \quad \mathbf{B+C+D+E+F} \\
 \begin{array}{|c|c|c|c|} \hline a & b & 0 & 0 \\ \hline c & b & 0 & c-b \\ \hline 0 & 0 & a & b \\ \hline 0 & 0 & c & c \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & d-b & 0 & b-c \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & d-c \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}
 \end{array}$$

A=B+C+D+E+F

$$\begin{array}{|c|c|c|c|} \hline a & b & 0 & 0 \\ \hline c & d & 0 & 0 \\ \hline 0 & 0 & a & b \\ \hline 0 & 0 & c & d \\ \hline \end{array}
 \begin{array}{|c|} \hline e \\ \hline f \\ \hline g \\ \hline h \\ \hline \end{array}
 =
 \begin{array}{|l} \hline ae+bf \\ \hline ce+df \\ \hline ag+bh \\ \hline cg+dh \\ \hline \end{array}$$

A little magic...

Claim: We can compute the products $Bv, Cv, Dv, Ev,$ and Fv with 7 scalar multiplications

So we can compute Av with 7 scalar multiplications

Bv: 1 multiplication

$$\begin{array}{|c|c|c|c|} \hline b & b & 0 & 0 \\ \hline b & b & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}
 \begin{array}{|c|} \hline e \\ \hline f \\ \hline g \\ \hline h \\ \hline \end{array}
 =
 \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

Cv: 1 multiplication (2 so far)

$$\begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & c & c \\ \hline 0 & 0 & c & c \\ \hline \end{array}
 \begin{array}{|c|} \hline e \\ \hline f \\ \hline g \\ \hline h \\ \hline \end{array}
 =
 \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

Dv: 1 multiplication (3 so far)

$$\begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline c-b & 0 & 0 & c-b \\ \hline b-c & 0 & 0 & b-c \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}
 \begin{array}{|c|} \hline e \\ \hline f \\ \hline g \\ \hline h \\ \hline \end{array}
 =
 \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

Ev: 2 multiplication (5 so far)

$$\begin{array}{|c|c|c|c|} \hline \text{E} & & & \\ \hline \text{a-b} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \text{c-b} & 0 & \text{a-c} & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}
 \quad
 \begin{array}{|c|} \hline \text{v} \\ \hline \text{e} \\ \hline \text{f} \\ \hline \text{g} \\ \hline \text{h} \\ \hline \end{array}
 =
 \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

Note: $(c-b)e+(a-c)g=(a-c)(g-e)+(a-b)e$

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Fv: 2 multiplication (7total)

$$\begin{array}{|c|c|c|c|} \hline \text{F} & & & \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & \text{d-b} & 0 & \text{b-c} \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \text{d-c} \\ \hline \end{array}
 \quad
 \begin{array}{|c|} \hline \text{v} \\ \hline \text{e} \\ \hline \text{f} \\ \hline \text{g} \\ \hline \text{h} \\ \hline \end{array}
 =
 \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

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Strassen's Divide and Conquer

• Input:

$$\begin{array}{|c|c|} \hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|} \hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline \end{array}$$

• Output:

$$\begin{array}{|c|c|} \hline C_{11} & C_{12} \\ \hline C_{21} & C_{22} \\ \hline \end{array}$$

$$C_{ij} = A_{1i}B_{1j} + A_{2i}B_{2j}$$

$$T(2n^2) = 7T(2(n/2)^2) + cn^2 = O(n^{\log_3 7})$$

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