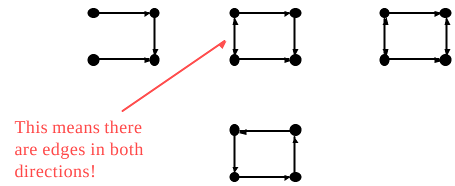
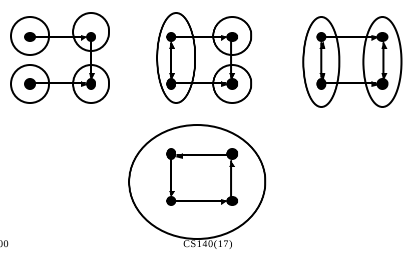
Digraph notions

- Vertex y is *reachable* from x if there is a directed path in G from x to y.
  (By convention x is reachable from x by a directed path of length 0.)
- Vertices x and y are *strongly connected* if x is reachable from y and y is reachable from x.
- Vertices x and y are *weakly connected* if they are in the same connected component of the undirected version of G.

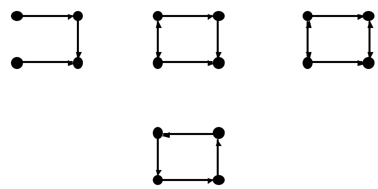
Strongly-connected vertices?



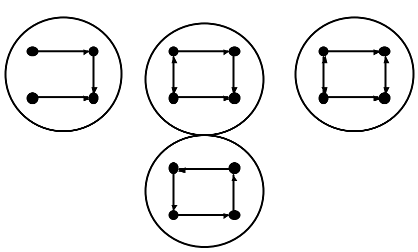
Strongly-connected vertices:



Weakly-connected vertices?



Weakly-connected vertices:



Digraph notions cont.

- Strongly connected is an equivalence relation; the equivalence classes are called *strongly connected components*.
- Weakly connected is an equivalence relation; the equivalence classes are called *weakly connected components*.

DFS Applications

- Identify the strongly connected components of a digraph G.
- Identify the weakly connected components of a digraph G. Easy! Do on your own.

DFS(G)

DFS(G)
  While G has an unvisited vertex x:
    DFS(x)

We'll choose these in alphabetical order

Note: DFS is overloaded!

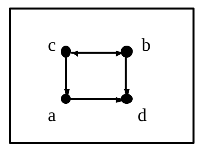
DFS(**x**)

DFS(x)
  Mark x visited
  For each edge <x,y>
    If y is unvisited then
      DFS(y)

We'll choose these in alphabetical order

DFS(G)
Alphabetical priority

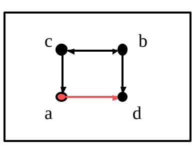
a is unvisited so DFS(a)



Call Stack:
DFS(G)

DFS(a)
Alphabetical priority

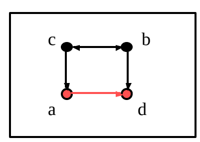
Visit a
Find <a,d> edge and call DFS(d)



Call Stack:
DFS(a)
DFS(G)

DFS(d)
Alphabetical priority

Visit d
All out-edges checked so return



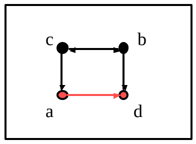
Call Stack:
DFS(d)
DFS(a)
DFS(G)

DFS(a)
Alphabetical priority

Visit a

Find <a,d> edge and call DFS(d)

All out-edges checked so return

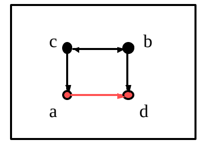


Call Stack:
DFS(a)
DFS(G)

DFS(G)
Alphabetical priority

a is unvisited so DFS(a)

b is unvisited so DFS(b)

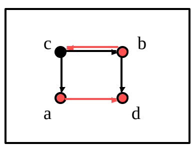


Call Stack:
DFS(G)

DFS(b)
Alphabetical priority

Visit b

Find edge <b,c> and call DFS(c)



Call Stack:
DFS(b)
DFS(G)

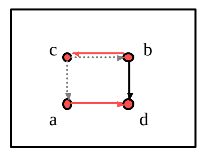
DFS(c)
Alphabetical priority

Visit c

Find edge <c,a> – no action

Find edge <c,b> – no action

All out-edges checked so return



Call Stack:
DFS(c)
DFS(b)
DFS(G)

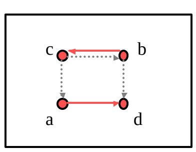
DFS(b)
Alphabetical priority

Visit b

Find edge <b,c> and call DFS(c)

Find edge <b,d> – no action

All out-edges checked so return



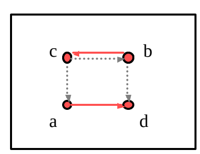
Call Stack:
DFS(b)
DFS(G)

DFS(G)
Alphabetical priority

a is unvisited so DFS(a)

b is unvisited so DFS(b)

All nodes checked so return



Call Stack:
DFS(G)

What is the running time of DFS?

- O(m+n)

- Every vertex is pushed onto the stack once and popped from the stack once.
- Each out-edge is inspected once.

SCC Problem

- Input: Digraph G
- Output: The strongly connected components of G.

Naïve Algorithm

For each pair of vertices x and y:

If x is reachable from y and y is reachable from x then x and y are strongly connected.

Run DFS

Naïve Algorithm

For each pair of vertices x and y:

If x is reachable from y and y is reachable from x then x and y are strongly connected.

Mark the vertices unvisited
DFS(y)
If x is marked visited then x is reachable from y
Else x is not reachable from y

Naïve algorithm

- Worst case: _______ calls to DFS(x) so the running time is ____________

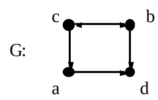
All little more sophistication please…

- We can find the strongly connected components of G with two calls to DFS(G)
- Three ideas
  - DFS Forest
  - Timestamps
  - Reversal of G

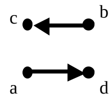
DFS Forest

- The DFS Forest of G is the subgraph consisting of
  - Every vertex of G
  - Each edge traversed in DFS(G)

Different selection rules give different results



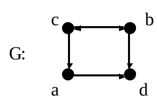
DFS tree of G:



WARNING

- DFS Forests are sometimes

DFS Forests: Enumerate

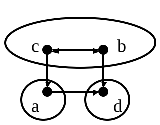


What can we say?

- If the DFS forest of G ________________ then x and y are strongly connected.
- If x and y are strongly connected then _____________

DFS Forest

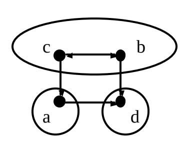
- If x and y are strongly connected then they are in the same tree of (every) DFS forest of G.



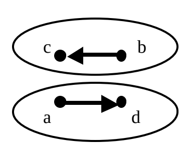
Strongly connected components of G

DFS Forest

Strongly-connected in G is a refinements of the weakly connected relation of any DFS forest of G



Strongly connected components of G



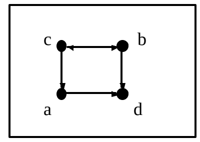
DFS forest of G

Three ideas

- DFS Forest of G
- **Timestamps**
- Reversal

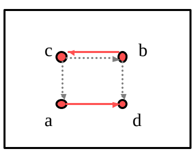
DFS(G)
Alphabetical order

Record first-arrival and last-departure times.



	First-arrival	Last-Departure
a		
b		
c		
d		

DFS(G)
Alphabetical order

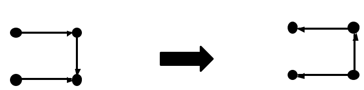


	First-arrival	Last-Departure
a	1	4
b	5	8
c	6	7
d	2	3

Three ideas

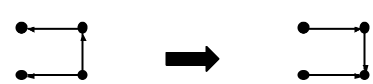
- DFS Forest
- Timestamps
- **Reversal of G**

G^R: Reverse the edges of G



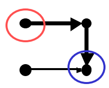
G $\quad$ G^R

$(G^R)^R$: Reverse the edges of G^R

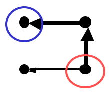


G^R $\quad$ $(G^R)^R=G$

Reachability



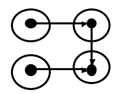
G



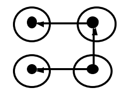
G^R

X is reachable from Y in G $\Leftrightarrow$ Y is reachable from X in G^T

SCC



G



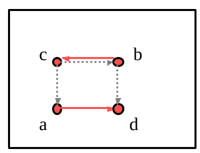
G^R

The Strongly Connected Components of G and G^R are the same!

SCC Algorithm

- DFS(G) with timestamp
- DFS(G^R) using last-departure time decreasing order to produce a DFS forest F of G^R
- If x and y are weakly in F if and only if they are strongly connected in G.

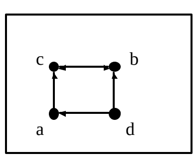
DFS(G)



	First-arrival	Last-Departure
a	1	4
b	5	8
c	6	7
d	2	3

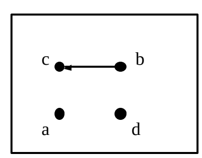
DFS(G^R)

Order: b,c,a,d



DFS Forest

Order: b,c,a,d



Correctness

Claim: If x and y are in the same strongly connected component of G if and only if x and y are in the same tree of the DFS forest of G^R.

Claim ⇒

If x and y are in the same strongly connected component of G then they are in the same tree of the DFS forest of G^R.

Claim ⇒

If x and y are in the same strongly connected component of G then they are in the same tree of the DFS forest of G^R.

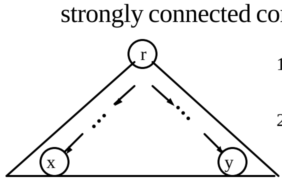
- If x and y are in the same SCC of G then x and y are in the same SCC of G^R.
- If x and y are in the same SCC of G^R then they are in the same tree of the DFS forest of G^R.

Claim ⇐

If x and y are in the same tree of the DFS forest of G^R then they are in the same strongly connected component of G.

Claim ⇐

If x and y are in the same tree of the DFS forest of G^R then they are in the same strongly connected component of G.



1. Show x and r are strongly connected in G.
2. Show y and r are strongly connected in G.

Claim ⇐

x and r are strongly connected in G

1. Since x is reachable from r in G^R, r is reachable from x in G.
2. Since ___________________, x is reachable from r in G.

Notice

- Last-departure(x) < Last-departure(r).
- If Last-departure (x) < First-arrival(r) then r is not reachable from x in G ⇒⇐. So Last-departure(x) > First-arrival(r)
- Hence

First-arrival(r)<First-arrival(x) <
Last-departure(x) < Last-departure (r)

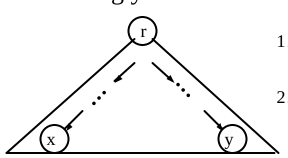
Claim ⇐

x and r are strongly connected in G

1. Since x is reachable from r in G^R, r is reachable from x in G.
2. Since First-arrival(r)<First-arrival(x)<Last-departure(x)<Last-departure(r), x is reachable from r in G.

Claim ⇐

If x and y are in the same tree of the DFS forest of G^R then they are in the same strongly connected component of G.



DONE

1. Show x and r are strongly connected in G.
2. Show y and r are strongly connected in G.

Same Argument

Correctness

Claim: If x and y are in the same strongly connected component of G if and only if x and y are in the same tree of the DFS forest of G^R.

DONE

Running Time

- We can find the strongly connected components of G in _______________
  - How do we represent G?
  - Do we have to sort the vertices by last-departure time?