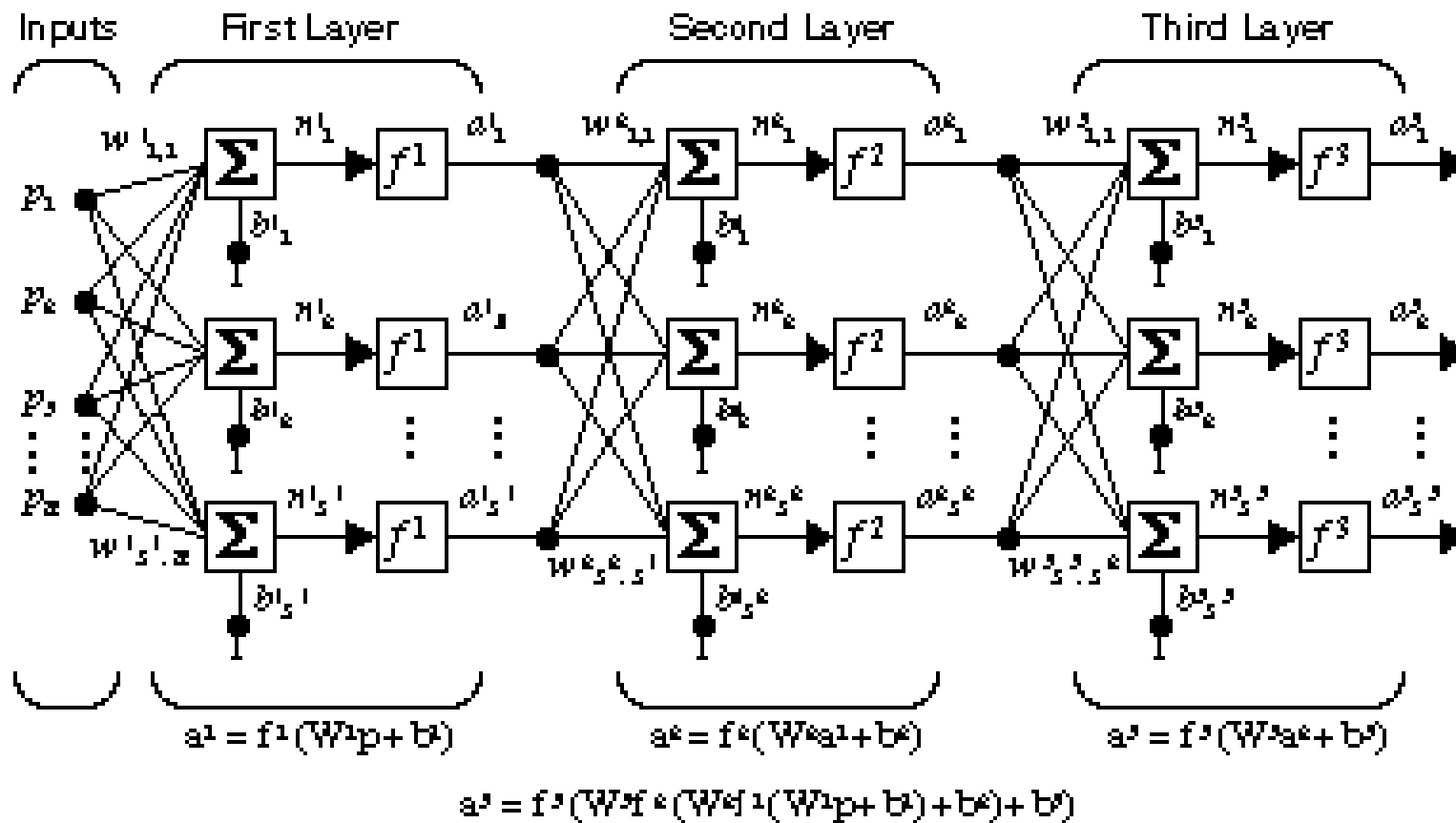
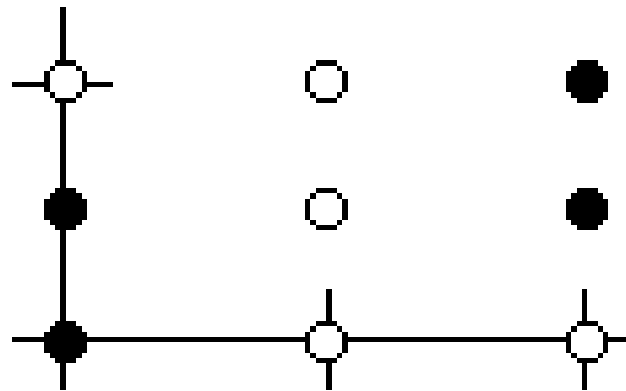


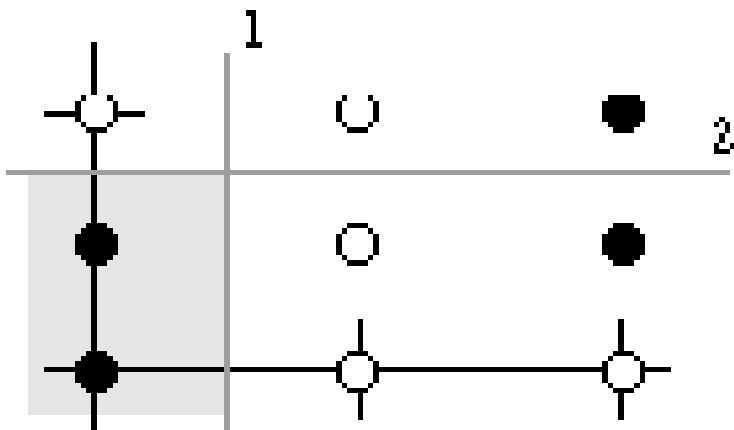


Backpropagation



R - S¹ - S² - S³ Network





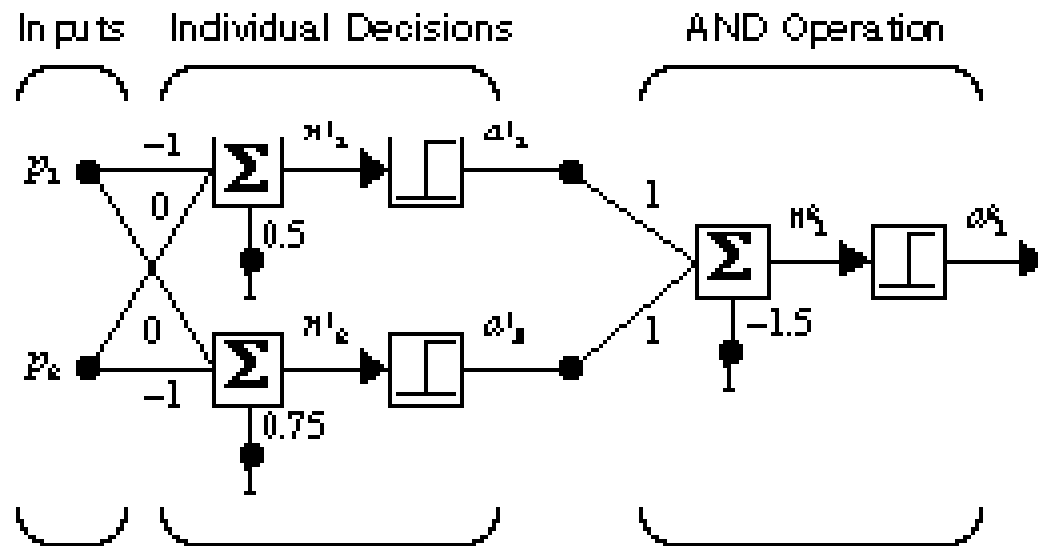
First Boundary:

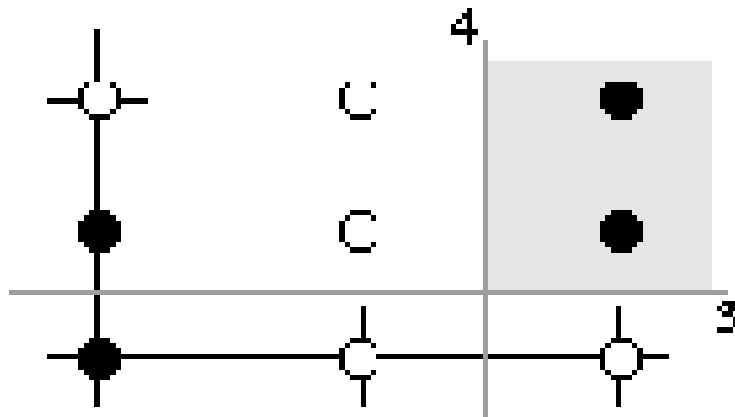
$$a_1^1 = \text{hardlim} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{p} + 0.5 \right)$$

Second Boundary:

$$a_2^1 = \text{hardlim} \left(\begin{bmatrix} 0 & -1 \end{bmatrix} \mathbf{p} + 0.75 \right)$$

First Subnetwork





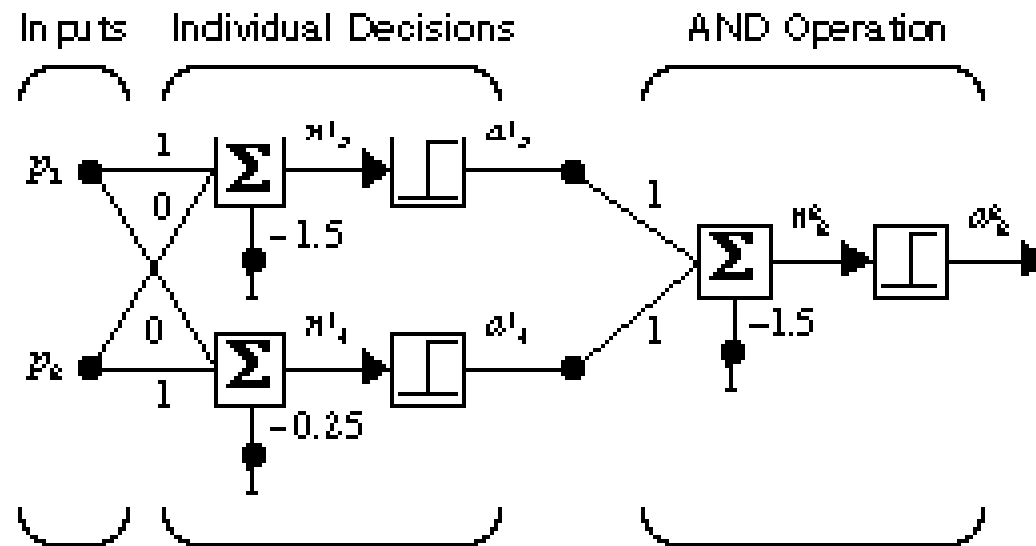
Third Boundary:

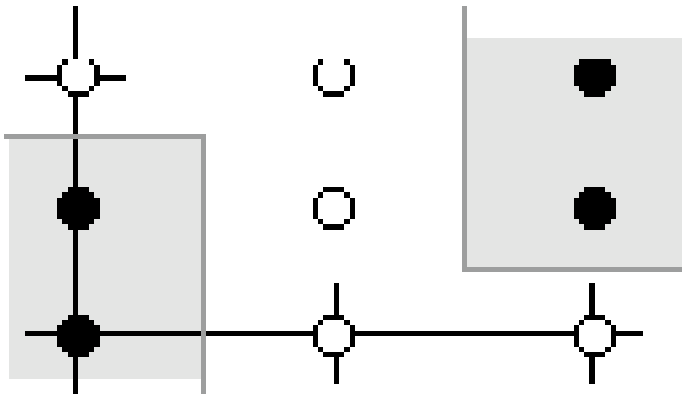
$$a_3^1 = \text{hardlim}([1 \ 0]\mathbf{p} - 1.5)$$

Fourth Boundary:

$$a_4^1 = \text{hardlim}([0 \ 1]\mathbf{p} - 0.25)$$

Second Subnetwork





$$W^1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$b^1 = \begin{bmatrix} 0.5 \\ 0.75 \\ -1.5 \\ -0.25 \end{bmatrix}$$

$$W^2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b^2 = \begin{bmatrix} -1.5 \\ -1.5 \end{bmatrix}$$

$$W^3 = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

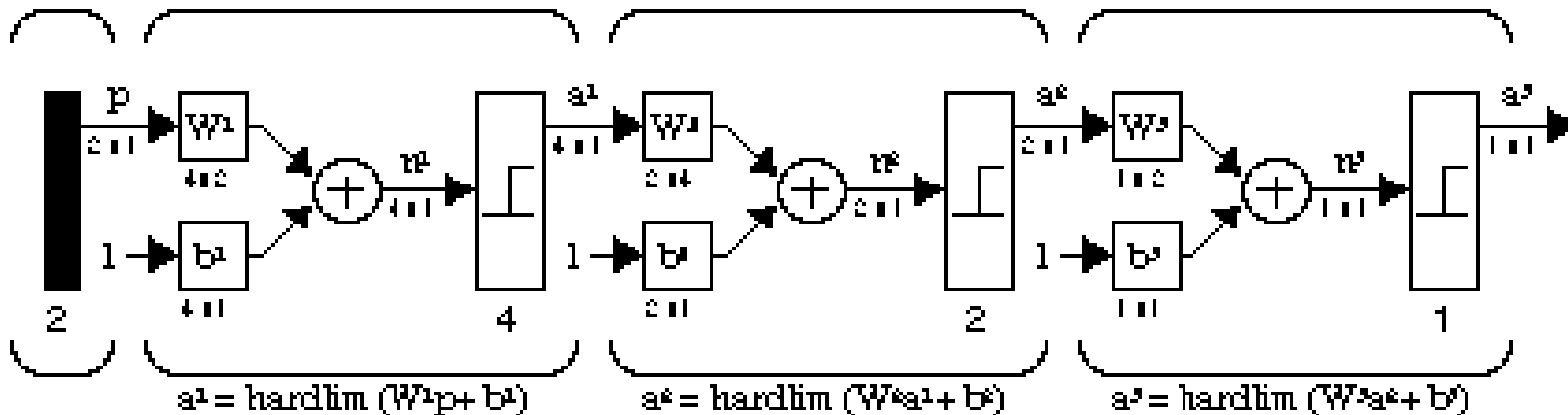
$$b^3 = \begin{bmatrix} -0.5 \end{bmatrix}$$

Input

Initial Decisions

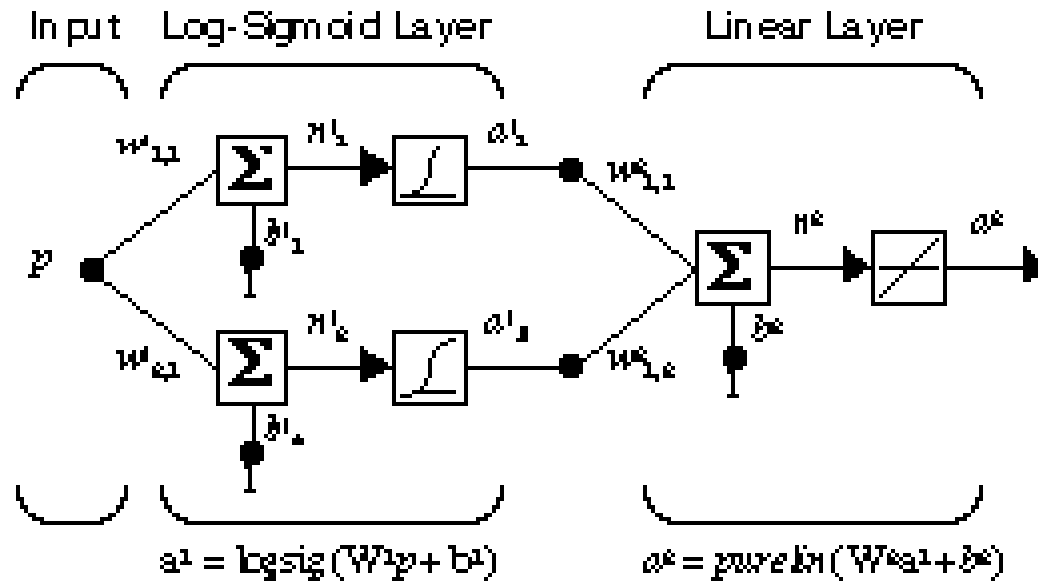
AND Operations

OR Operation



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Function Approximation Example



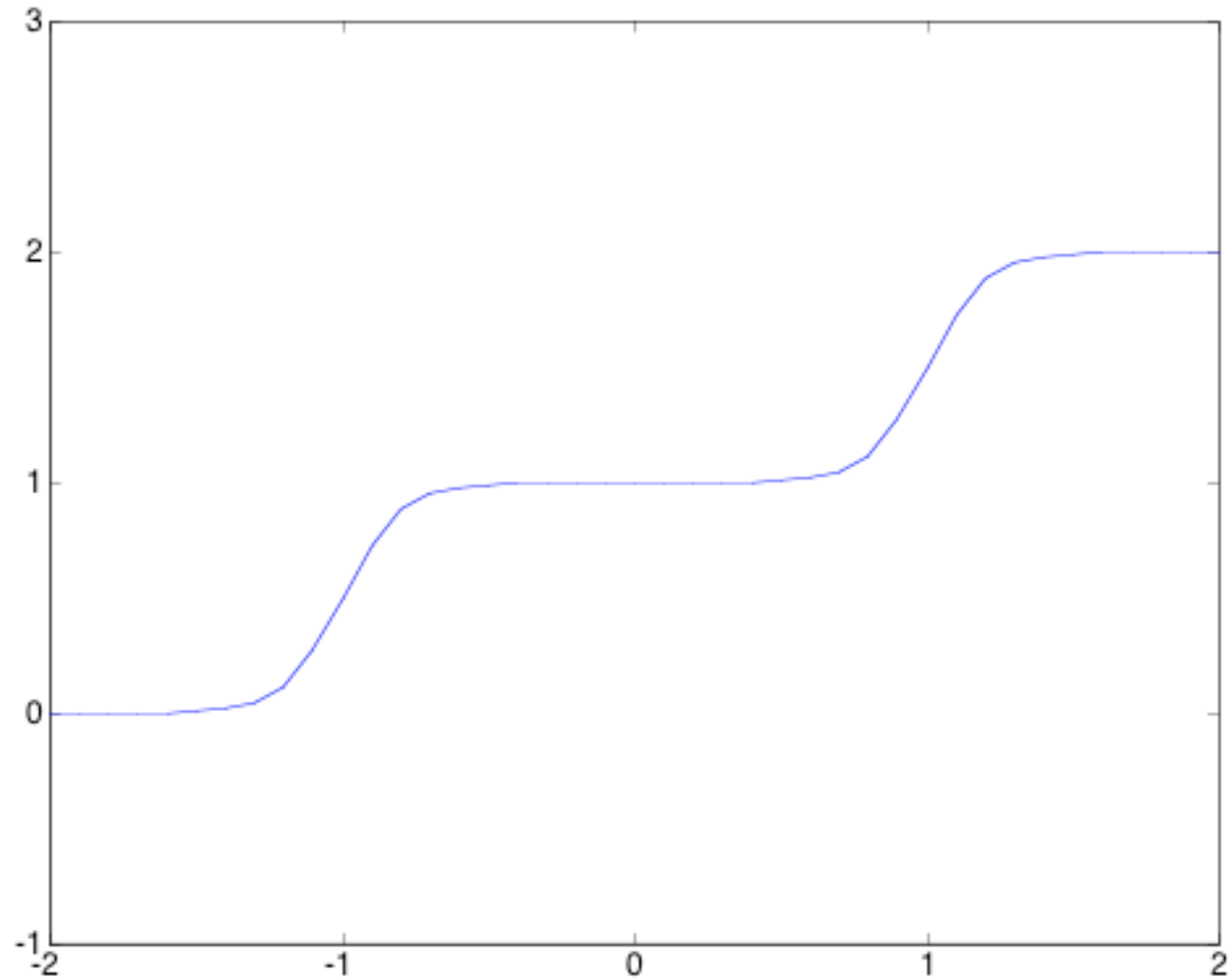
$$f^1(n) = \frac{1}{1 + e^{-n}}$$

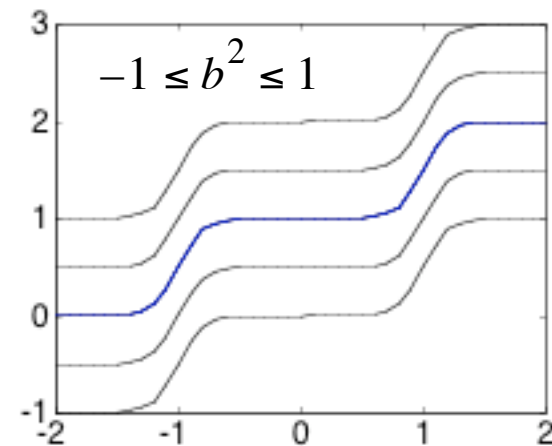
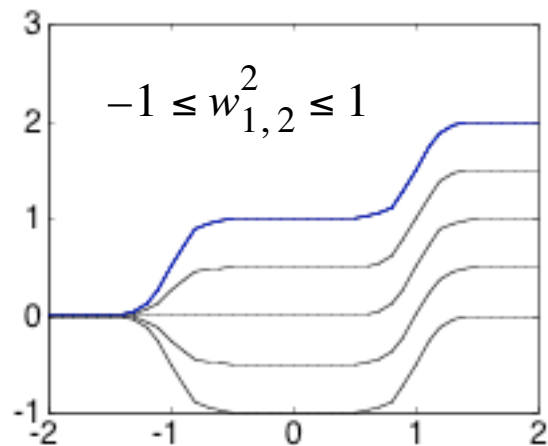
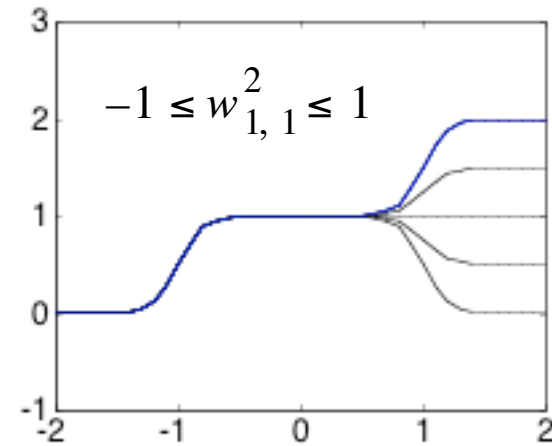
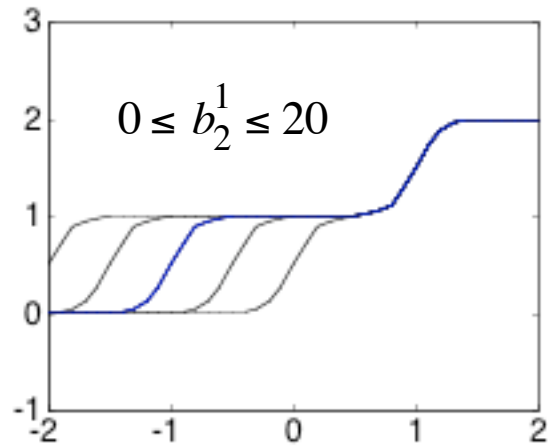
$$f^2(n) = n$$

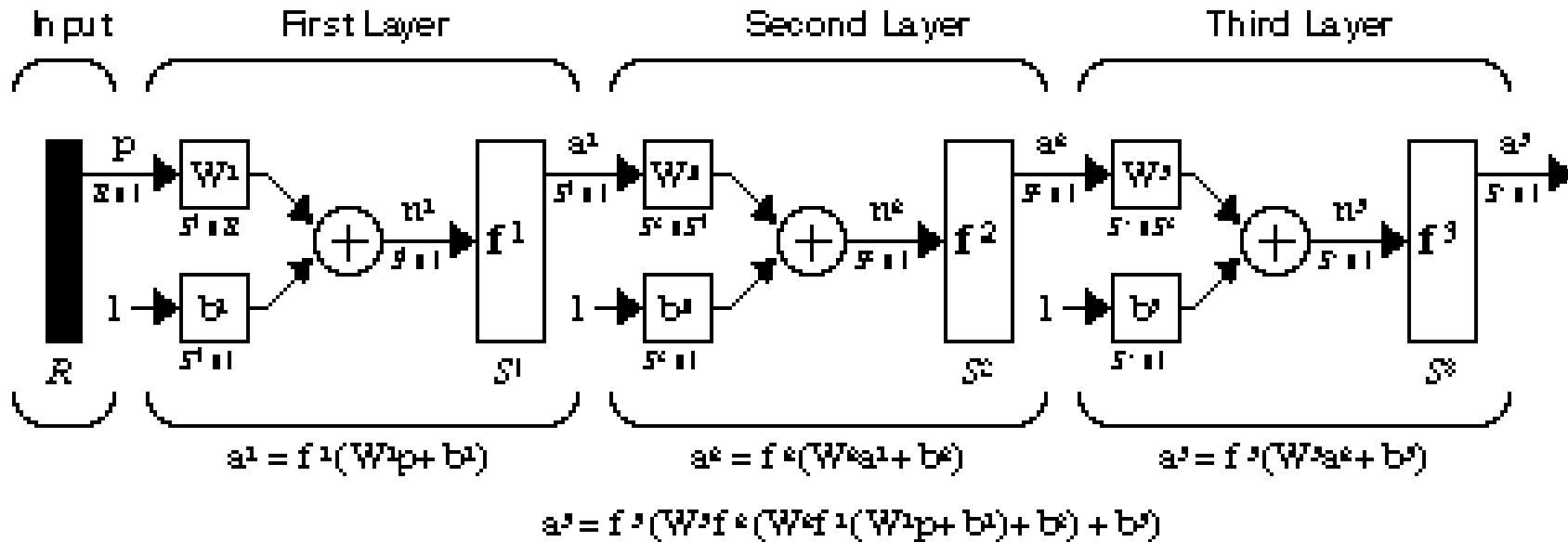
Nominal Parameter Values

$$w^1_{1,1} = 10 \quad w^1_{2,1} = 10 \quad b^1_1 = -10 \quad b^1_2 = 10$$

$$w^2_{1,1} = 1 \quad w^2_{1,2} = 1 \quad b^2 = 0$$







$$\mathbf{a}^{m+1} = \mathbf{f}^{m+1}(\mathbf{W}^{m+1} \mathbf{a}^m + \mathbf{b}^{m+1}) \quad m = 0, 2, \dots, M-1$$

$$\mathbf{a}^0 = \mathbf{p}$$

$$\mathbf{a} = \mathbf{a}^M$$



Training Set

$$\{\mathbf{p}_1, \mathbf{t}_1\}, \{\mathbf{p}_2, \mathbf{t}_2\}, \dots, \{\mathbf{p}_Q, \mathbf{t}_Q\}$$

Mean Square Error

$$F(\mathbf{x}) = E[e^2] = E[(t - a)^2]$$

Vector Case

$$F(\mathbf{x}) = E[\mathbf{e}^T \mathbf{e}] = E[(\mathbf{t} - \mathbf{a})^T (\mathbf{t} - \mathbf{a})]$$

Approximate Mean Square Error (Single Sample)

$$\hat{F}(\mathbf{x}) = (\mathbf{t}(k) - \mathbf{a}(k))^T (\mathbf{t}(k) - \mathbf{a}(k)) = \mathbf{e}^T(k) \mathbf{e}(k)$$

Approximate Steepest Descent

$$w_{i,j}^m(k+1) = w_{i,j}^m(k) - \alpha \frac{\partial \hat{F}}{\partial w_{i,j}^m} \quad b_i^m(k+1) = b_i^m(k) - \alpha \frac{\partial \hat{F}}{\partial b_i^m}$$



$$\frac{df(n(w))}{dw} = \frac{df(n)}{dn} \times \frac{dn(w)}{dw}$$

Example

$$f(n) = \cos(n) \quad n = e^{2w} \quad f(n(w)) = \cos(e^{2w})$$

$$\frac{df(n(w))}{dw} = \frac{df(n)}{dn} \times \frac{dn(w)}{dw} = (-\sin(n))(2e^{2w}) = (-\sin(e^{2w}))(2e^{2w})$$

Application to Gradient Calculation

$$\frac{\partial \hat{F}}{\partial w_{i,j}^m} = \frac{\partial \hat{F}}{\partial n_i^m} \times \frac{\partial n_i^m}{\partial w_{i,j}^m} \quad \frac{\partial \hat{F}}{\partial b_i^m} = \frac{\partial \hat{F}}{\partial n_i^m} \times \frac{\partial n_i^m}{\partial b_i^m}$$



$$n_i^m = \sum_{j=1}^{S^{m-1}} w_{i,j}^m a_j^{m-1} + b_i^m$$

$$\frac{\partial n_i^m}{\partial w_{i,j}^m} = a_j^{m-1} \qquad \frac{\partial n_i^m}{\partial b_i^m} = 1$$

Sensitivity

$$s_i^m \equiv \frac{\partial \hat{F}}{\partial n_i^m}$$

Gradient

$$\frac{\partial \hat{F}}{\partial w_{i,j}^m} = s_i^m a_j^{m-1} \qquad \frac{\partial \hat{F}}{\partial b_i^m} = s_i^m$$



$$w_{i,j}^m(k+1) = w_{i,j}^m(k) - \alpha s_i^m a_j^{m-1} \quad b_i^m(k+1) = b_i^m(k) - \alpha s_i^m$$

$$\mathbf{W}^m(k+1) = \mathbf{W}^m(k) - \alpha \mathbf{s}^m (\mathbf{a}^{m-1})^T \quad \mathbf{b}^m(k+1) = \mathbf{b}^m(k) - \alpha \mathbf{s}^m$$

$$\mathbf{s}^m \equiv \frac{\partial \hat{F}}{\partial \mathbf{n}^m} = \begin{bmatrix} \frac{\partial \hat{F}}{\partial n_1^m} \\ \frac{\partial \hat{F}}{\partial n_2^m} \\ \vdots \\ \frac{\partial \hat{F}}{\partial n_{S^m}^m} \end{bmatrix}$$

Next Step: Compute the Sensitivities (Backpropagation)



$$\frac{\partial \mathbf{n}^{m+1}}{\partial \mathbf{n}^m} \equiv \begin{bmatrix} \frac{\partial n_1^{m+1}}{\partial n_1^m} & \frac{\partial n_1^{m+1}}{\partial n_2^m} & \dots & \frac{\partial n_1^{m+1}}{\partial n_{S^m}^m} \\ \frac{\partial n_2^{m+1}}{\partial n_1^m} & \frac{\partial n_2^{m+1}}{\partial n_2^m} & \dots & \frac{\partial n_2^{m+1}}{\partial n_{S^m}^m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial n_{S^{m+1}}^{m+1}}{\partial n_1^m} & \frac{\partial n_{S^{m+1}}^{m+1}}{\partial n_2^m} & \dots & \frac{\partial n_{S^{m+1}}^{m+1}}{\partial n_{S^m}^m} \end{bmatrix}$$

$$\frac{\partial \mathbf{n}^{m+1}}{\partial \mathbf{n}^m} = \mathbf{W}^{m+1} \mathbf{F}^m(\mathbf{n}^m)$$

$$\frac{\partial n_i^{m+1}}{\partial n_j^m} = \frac{\partial \left(\sum_{l=1}^{S^m} w_{i,l}^{m+1} a_l^m + b_i^{m+1} \right)}{\partial n_j^m} = w_{i,j}^{m+1} \frac{\partial a_j^m}{\partial n_j^m}$$

$$\frac{\partial n_i^{m+1}}{\partial n_j^m} = w_{i,j}^{m+1} \frac{\partial f^m(n_j^m)}{\partial n_j^m} = w_{i,j}^{m+1} f^{\cdot m}(n_j^m)$$

$$f^{\cdot m}(n_j^m) = \frac{\partial f^m(n_j^m)}{\partial n_j^m}$$

$$\mathbf{F}^m(\mathbf{n}^m) = \begin{bmatrix} f^{\cdot m}(n_1^m) & 0 & \dots & 0 \\ 0 & f^{\cdot m}(n_2^m) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & f^{\cdot m}(n_{S^m}^m) \end{bmatrix}$$



$$\mathbf{s}^m = \frac{\partial \hat{F}}{\partial \mathbf{n}^m} = \left(\frac{\partial \mathbf{n}^{m+1}}{\partial \mathbf{n}^m} \right)^T \frac{\partial \hat{F}}{\partial \mathbf{n}^{m+1}} = \mathbf{F}^m(\mathbf{n}^m) (\mathbf{W}^{m+1})^T \frac{\partial \hat{F}}{\partial \mathbf{n}^{m+1}}$$

$$\mathbf{s}^m = \mathbf{F}^m(\mathbf{n}^m) (\mathbf{W}^{m+1})^T \mathbf{s}^{m+1}$$

The sensitivities are computed by starting at the last layer, and then propagating backwards through the network to the first layer.

$$\mathbf{s}^M \rightarrow \mathbf{s}^{M-1} \rightarrow \dots \rightarrow \mathbf{s}^2 \rightarrow \mathbf{s}^1$$



$$s_i^M = \frac{\partial \hat{F}}{\partial n_i^M} = \frac{\partial (\mathbf{t} - \mathbf{a})^T (\mathbf{t} - \mathbf{a})}{\partial n_i^M} = \frac{\partial \sum_{j=1}^{S^M} (t_j - a_j)^2}{\partial n_i^M} = -2(t_i - a_i) \frac{\partial a_i}{\partial n_i^M}$$

$$\frac{\partial a_i}{\partial n_i^M} = \frac{\partial a_i^M}{\partial n_i^M} = \frac{\partial f^M(n_i^M)}{\partial n_i^M} = f^M(n_i^M)$$

$$s_i^M = -2(t_i - a_i) f^M(n_i^M)$$

$$\mathbf{s}^M = -2\mathbf{F}^M(\mathbf{n}^M)(\mathbf{t} - \mathbf{a})$$



Forward Propagation

$$\mathbf{a}^0 = \mathbf{p}$$

$$\mathbf{a}^{m+1} = \mathbf{f}^{m+1}(\mathbf{W}^{m+1} \mathbf{a}^m + \mathbf{b}^{m+1}) \quad m = 0, 2, \dots, M-1$$

$$\mathbf{a} = \mathbf{a}^M$$

Backpropagation

$$\mathbf{s}^M = -2\dot{\mathbf{F}}^M(\mathbf{n}^M)(\mathbf{t} - \mathbf{a})$$

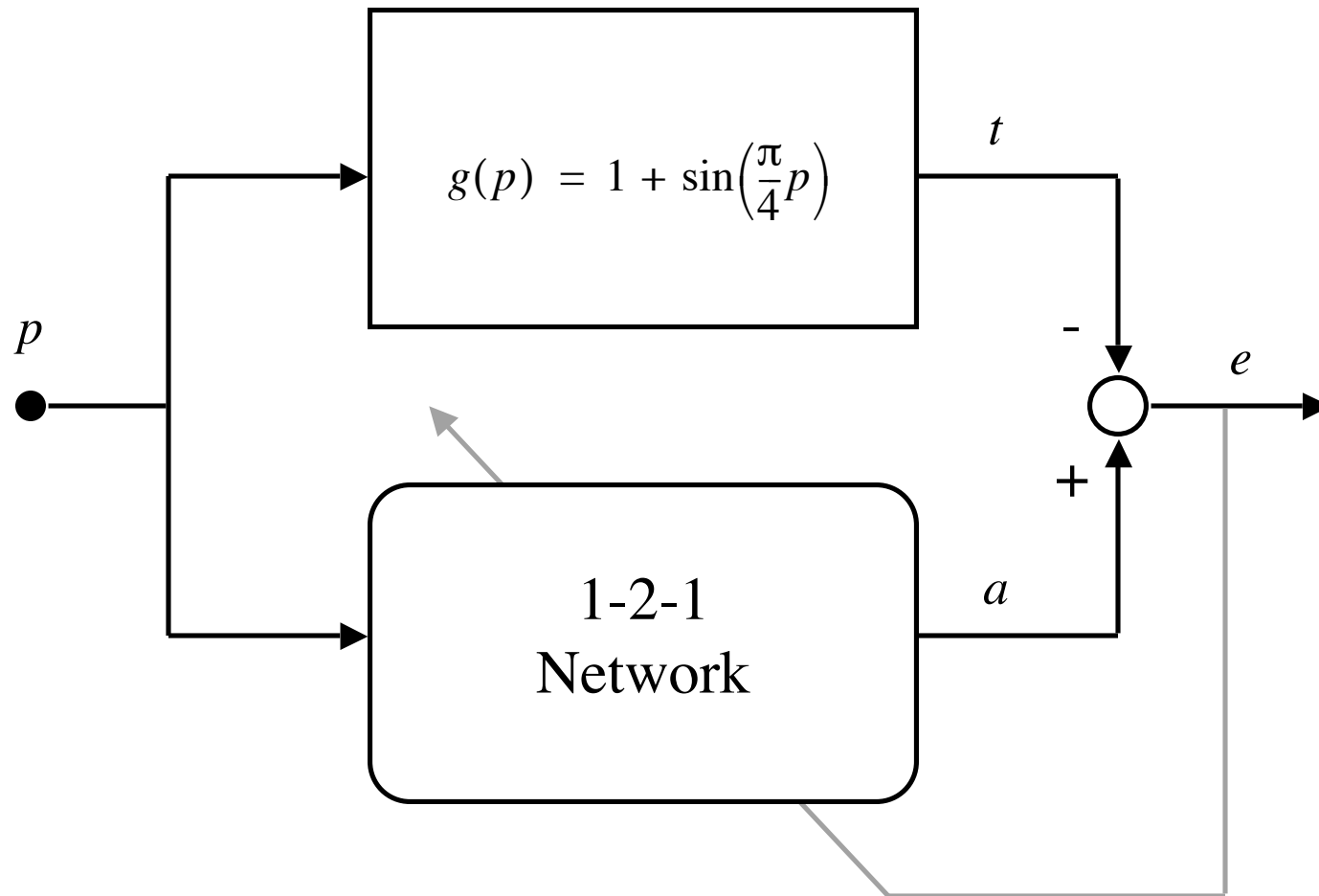
$$\mathbf{s}^m = \dot{\mathbf{F}}^m(\mathbf{n}^m)(\mathbf{W}^{m+1})^T \mathbf{s}^{m+1} \quad m = M-1, \dots, 2, 1$$

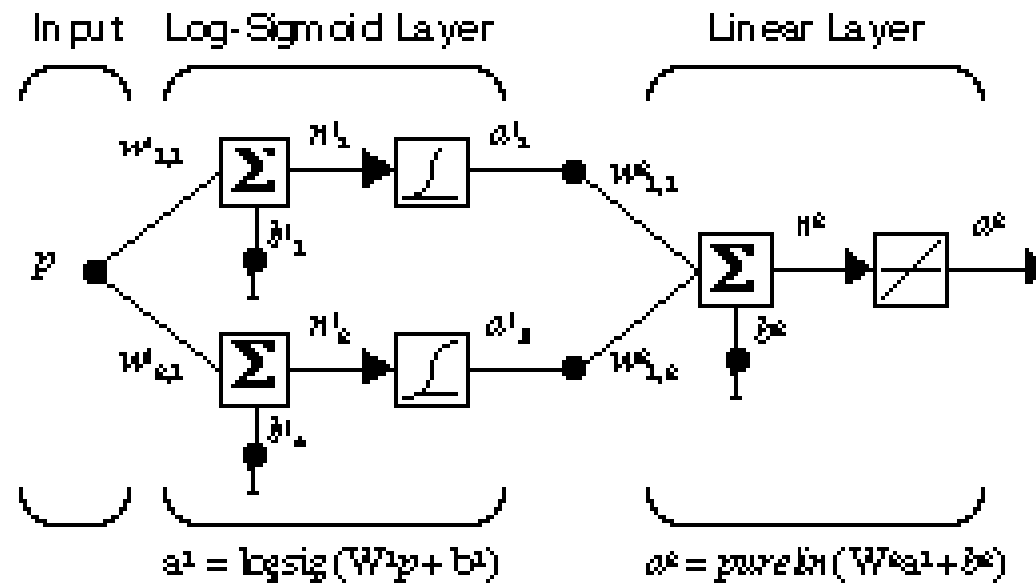
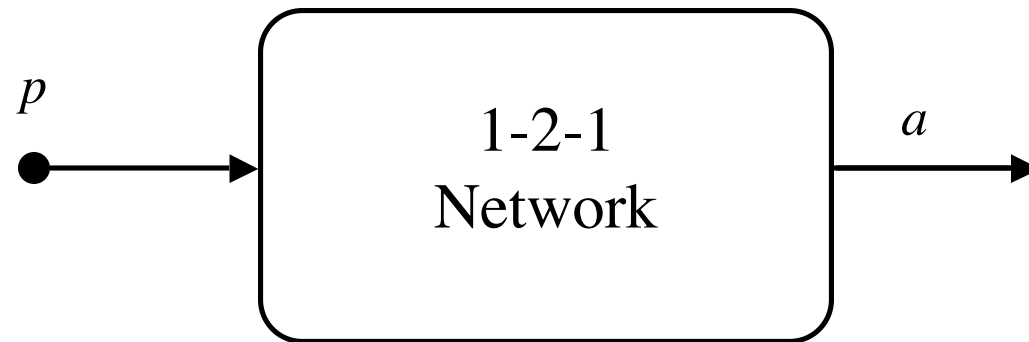
Weight Update

$$\mathbf{W}^m(k+1) = \mathbf{W}^m(k) - \alpha \mathbf{s}^m (\mathbf{a}^{m-1})^T \quad \mathbf{b}^m(k+1) = \mathbf{b}^m(k) - \alpha \mathbf{s}^m$$

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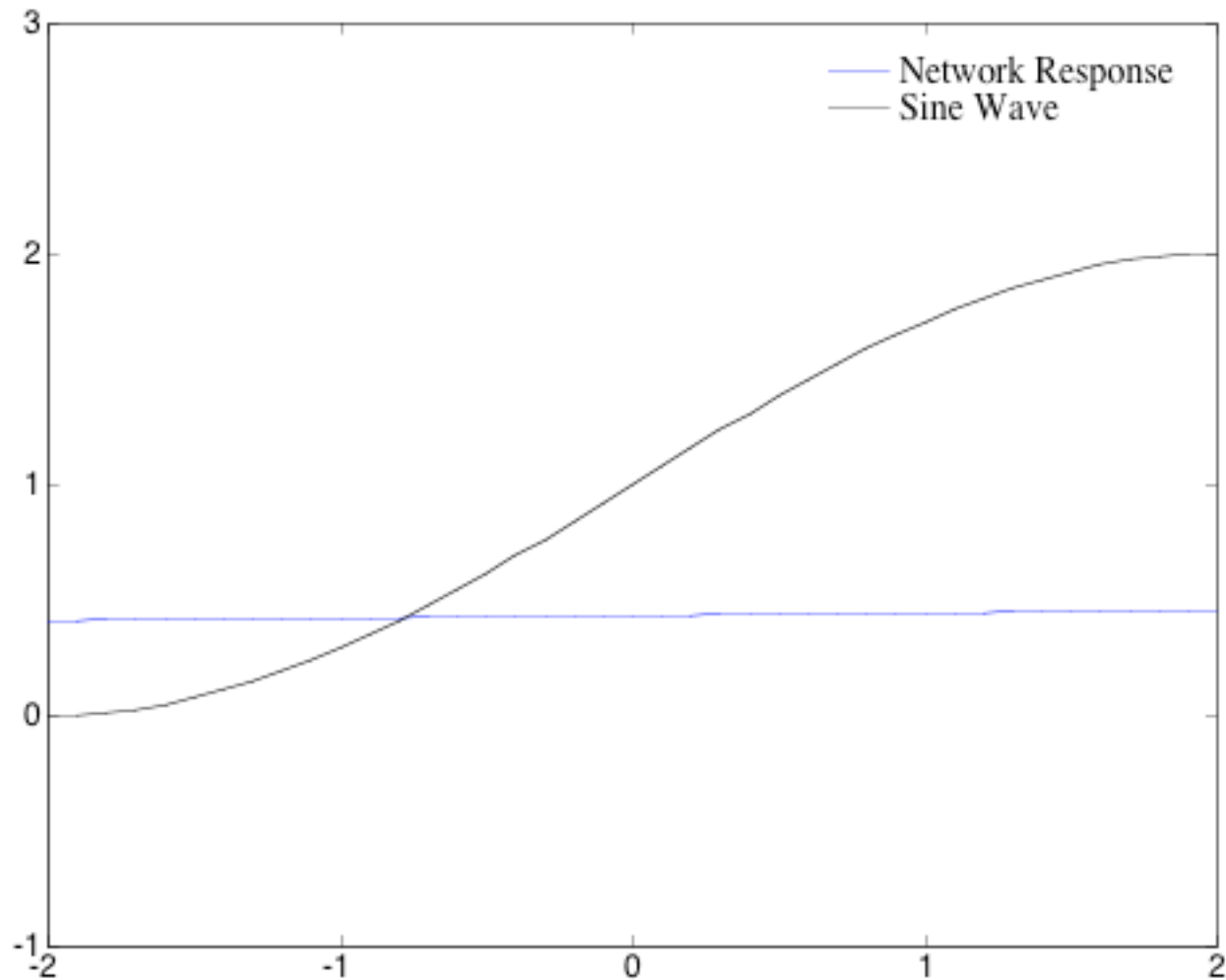
Example: Function Approximation







$$\mathbf{w}^1(0) = \begin{bmatrix} -0.27 \\ -0.41 \end{bmatrix} \quad \mathbf{b}^1(0) = \begin{bmatrix} -0.48 \\ -0.13 \end{bmatrix} \quad \mathbf{w}^2(0) = [0.09 \ -0.17] \quad \mathbf{b}^2(0) = [0.48]$$





$$a^0 = p = 1$$

$$\mathbf{a}^1 = \mathbf{f}^1(\mathbf{W}^1 \mathbf{a}^0 + \mathbf{b}^1) = \mathbf{log\,sig} \left(\begin{bmatrix} -0.27 \\ -0.41 \end{bmatrix} [1] + \begin{bmatrix} -0.48 \\ -0.13 \end{bmatrix} \right) = \mathbf{log\,sig} \left(\begin{bmatrix} -0.75 \\ -0.54 \end{bmatrix} \right)$$

$$\mathbf{a}^1 = \begin{bmatrix} \frac{1}{1 + e^{0.75}} \\ \frac{1}{1 + e^{0.54}} \end{bmatrix} = \begin{bmatrix} 0.321 \\ 0.368 \end{bmatrix}$$

$$a^2 = \mathbf{f}^2(\mathbf{W}^2 \mathbf{a}^1 + \mathbf{b}^2) = \mathit{purelin} \left(\begin{bmatrix} 0.09 & -0.17 \end{bmatrix} \begin{bmatrix} 0.321 \\ 0.368 \end{bmatrix} + \begin{bmatrix} 0.48 \end{bmatrix} \right) = \begin{bmatrix} 0.446 \end{bmatrix}$$

$$e = t - a = \left\{ 1 + \sin\left(\frac{\pi}{4}p\right) \right\} - a^2 = \left\{ 1 + \sin\left(\frac{\pi}{4}1\right) \right\} - 0.446 = 1.261$$



$$f^1(n) = \frac{d}{dn} \left(\frac{1}{1 + e^{-n}} \right) = \frac{e^{-n}}{(1 + e^{-n})^2} = \left(1 - \frac{1}{1 + e^{-n}} \right) \left(\frac{1}{1 + e^{-n}} \right) = (1 - a^1)(a^1)$$

$$f^2(n) = \frac{d}{dn}(n) = 1$$



$$\mathbf{s}^2 = -2\mathbf{F}'^2(\mathbf{n}^2)(\mathbf{t} - \mathbf{a}) = -2\left[f'^2(n^2)\right](1.261) = -2\left[1\right](1.261) = -2.522$$

$$\mathbf{s}^1 = \mathbf{F}'^1(\mathbf{n}^1)(\mathbf{W}^2)^T \mathbf{s}^2 = \begin{bmatrix} (1 - a_1^1)(a_1^1) & 0 \\ 0 & (1 - a_2^1)(a_2^1) \end{bmatrix} \begin{bmatrix} 0.09 \\ -0.17 \end{bmatrix} \begin{bmatrix} -2.522 \end{bmatrix}$$

$$\mathbf{s}^1 = \begin{bmatrix} (1 - 0.321)(0.321) & 0 \\ 0 & (1 - 0.368)(0.368) \end{bmatrix} \begin{bmatrix} 0.09 \\ -0.17 \end{bmatrix} \begin{bmatrix} -2.522 \end{bmatrix}$$

$$\mathbf{s}^1 = \begin{bmatrix} 0.218 & 0 \\ 0 & 0.233 \end{bmatrix} \begin{bmatrix} -0.227 \\ 0.429 \end{bmatrix} = \begin{bmatrix} -0.0495 \\ 0.0997 \end{bmatrix}$$



$$\alpha = 0.1$$

$$\mathbf{W}^2(1) = \mathbf{W}^2(0) - \alpha \mathbf{s}^2 (\mathbf{a}^1)^T = \begin{bmatrix} 0.09 & -0.17 \end{bmatrix} - 0.1 \begin{bmatrix} -2.522 \end{bmatrix} \begin{bmatrix} 0.321 & 0.368 \end{bmatrix}$$

$$\mathbf{W}^2(1) = \begin{bmatrix} 0.171 & -0.0772 \end{bmatrix}$$

$$\mathbf{b}^2(1) = \mathbf{b}^2(0) - \alpha \mathbf{s}^2 = \begin{bmatrix} 0.48 \end{bmatrix} - 0.1 \begin{bmatrix} -2.522 \end{bmatrix} = \begin{bmatrix} 0.732 \end{bmatrix}$$

$$\mathbf{W}^1(1) = \mathbf{W}^1(0) - \alpha \mathbf{s}^1 (\mathbf{a}^0)^T = \begin{bmatrix} -0.27 \\ -0.41 \end{bmatrix} - 0.1 \begin{bmatrix} -0.0495 \\ 0.0997 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} -0.265 \\ -0.420 \end{bmatrix}$$

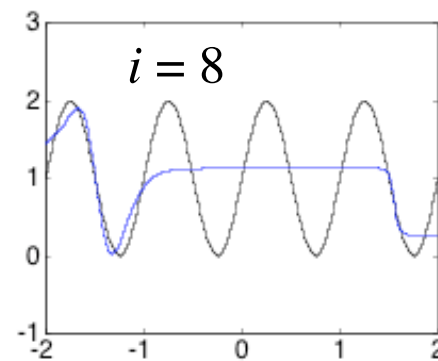
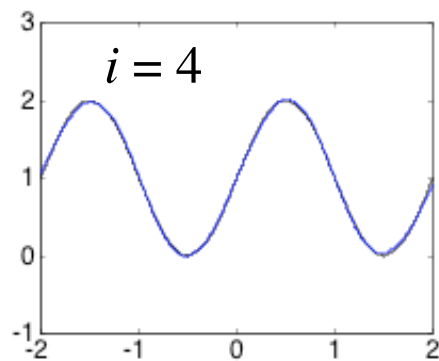
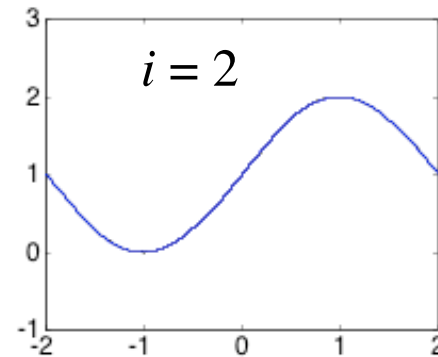
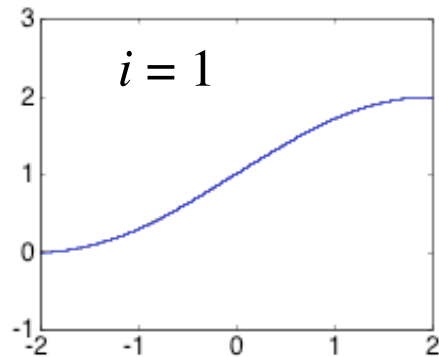
$$\mathbf{b}^1(1) = \mathbf{b}^1(0) - \alpha \mathbf{s}^1 = \begin{bmatrix} -0.48 \\ -0.13 \end{bmatrix} - 0.1 \begin{bmatrix} -0.0495 \\ 0.0997 \end{bmatrix} = \begin{bmatrix} -0.475 \\ -0.140 \end{bmatrix}$$

Choice of Architecture



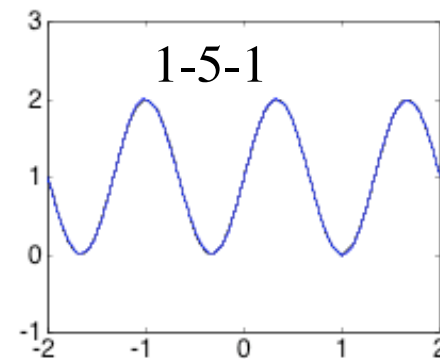
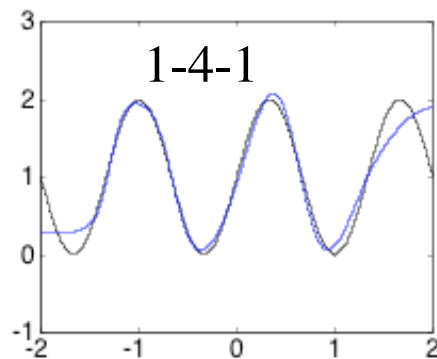
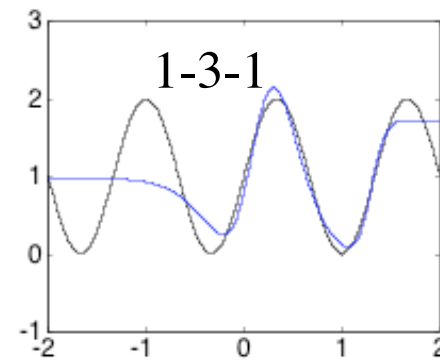
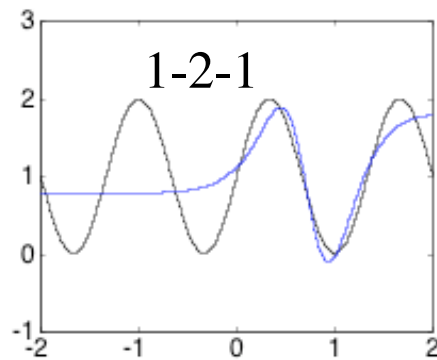
$$g(p) = 1 + \sin\left(\frac{i\pi}{4}p\right)$$

1-3-1 Network



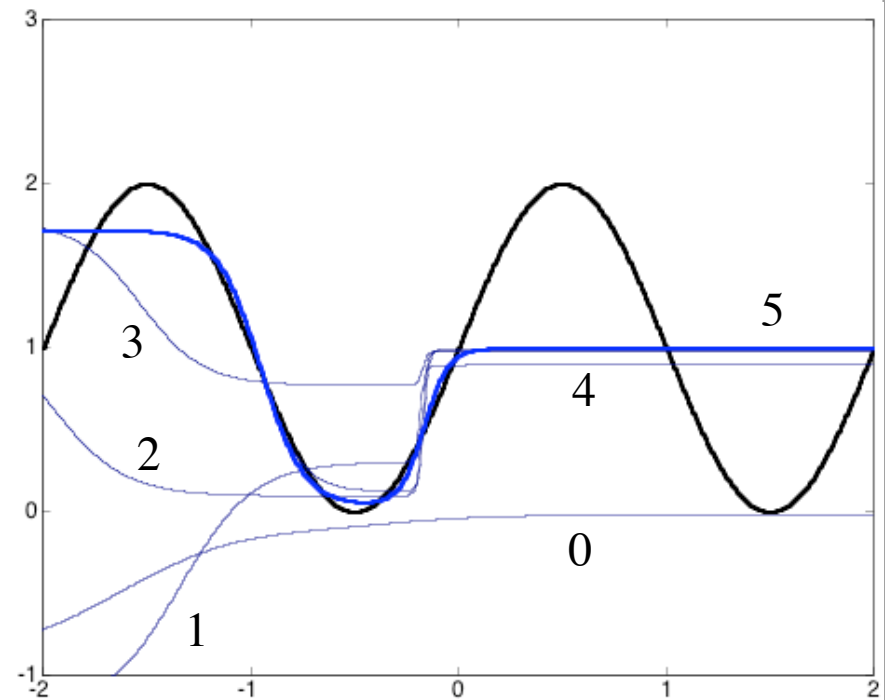
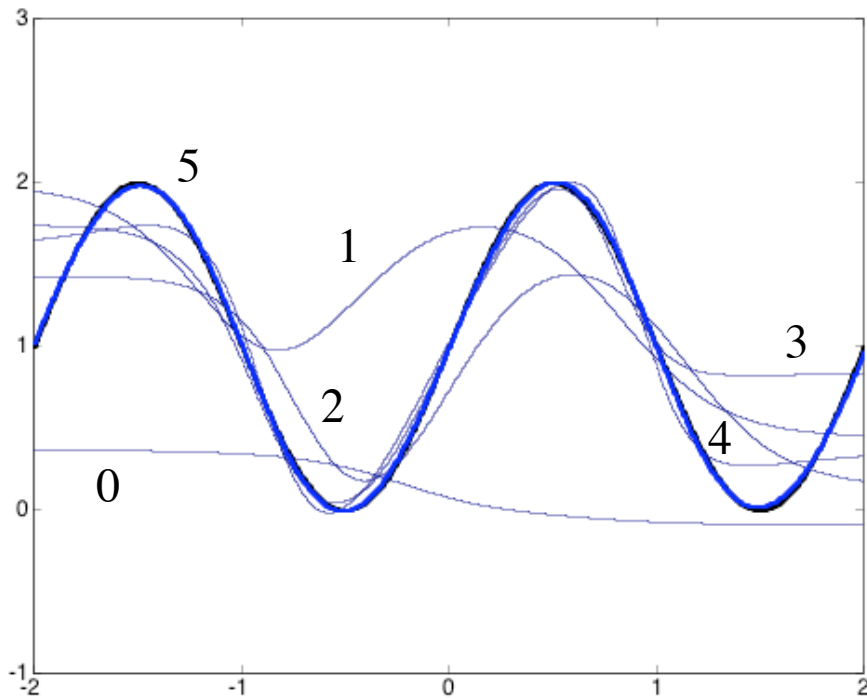


$$g(p) = 1 + \sin\left(\frac{6\pi}{4}p\right)$$





$$g(p) = 1 + \sin(\pi p)$$





$$\{\mathbf{p}_1, \mathbf{t}_1\}, \{\mathbf{p}_2, \mathbf{t}_2\}, \dots, \{\mathbf{p}_Q, \mathbf{t}_Q\}$$

$$g(p) = 1 + \sin\left(\frac{\pi}{4}p\right) \quad p = -2, -1.6, -1.2, \dots, 1.6, 2$$

