Associative Learning: Unsupervised Hebbian Learning

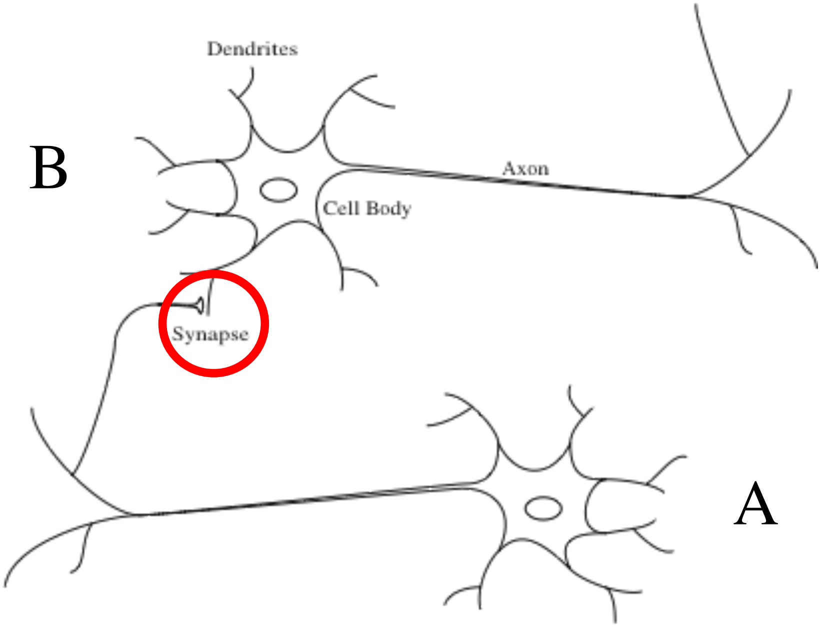
Anticipation

- William James, 1890: “When two brain processes are active together, or in immediate succession, one of them, on reoccurring, tends to propagate its excitement into the other.”

Hebb’s Postulate

“When an axon of cell A is near enough to excite a cell B and repeatedly or persistently takes part in firing it, some growth process or metabolic change takes place in one or both cells such that A’s efficiency, as one of the cells firing B, is increased.”

D. O. Hebb, 1949



Donald Hebb, 1904-1985

More from Hebb, 1949

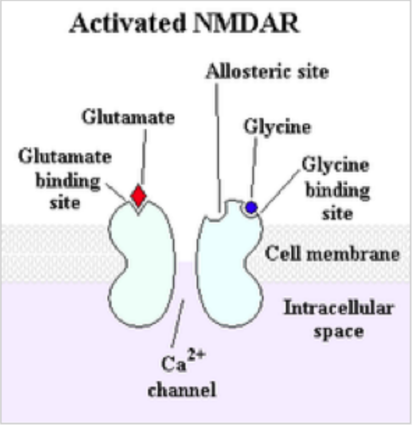
- "**The general idea is an old one**, that any two cells or systems of cells that are repeatedly active at the same time will tend to become 'associated', so that activity in one facilitates activity in the other." (Hebb 1949, p. 70)
- "When one cell repeatedly assists in firing another, the axon of the first cell develops **synaptic knobs** (or enlarges them if they already exist) in contact with the soma of the second cell." (Hebb 1949, p. 63)

Coincidence Detection

- Rojas, p. 21:
  The **NMDA** [N-methyl D-aspartate] **receptors** act as **coincidence detectors** of presynaptic and postsynaptic activity, which in turn leads to greater synaptic efficiency.

- Wikipedia:
  The NMDA receptor (NMDAR), a glutamate receptor, is **the predominant molecular device** for controlling synaptic plasticity and memory function.

NMDA Receptor (Wikipedia)



Stylised depiction of an activated NMDAR. Glutamate is in the glutamate binding site and glycine is in the glycine binding site. Allosteric sites that would cause inhibition of the receptor are not occupied. NMDARs require the binding of two molecules of glutamate or aspartate and two of glycine.[1]

The NMDA receptor forms a heterotetramer between two NR1 and two NR2 subunits; two obligatory NR1 subunits and two regionally localized NR2 subunits. A related gene family of NR3 A and B subunits have an inhibitory effect on receptor activity. Multiple receptor isoforms with distinct brain distributions and functional properties arise by selective splicing of the NR1 transcripts and differential expression of the NR2 subunits.

Each receptor subunit has modular design and each structural module also represents a functional unit:

- The *extracellular domain* contains two globular structures: a modulatory domain and a ligand-binding domain. NR1 subunits bind the co-agonist glycine and NR2 subunits bind the neurotransmitter glutamate.
- The *agonist-binding module* links to a membrane domain, which consists of three trans-membrane segments and a re-entrant loop reminiscent of the selectivity filter of potassium channels.
- The *membrane domain* contributes residues to the channel pore and is responsible for the receptor's high-unitary conductance, high-calcium permeability, and voltage-dependent magnesium block.
- Each subunit has an extensive *cytoplasmic domain*, which contain residues that can be directly modified by a series of protein kinases and protein phosphatases, as well as residues that interact with a large number of structural, adaptor, and scaffolding proteins.

The glycine-binding modules of the NR1 and NR3 subunits and the glutamate-binding module of the NR2A subunit have been expressed as soluble proteins, and their three-dimensional structure has been solved at atomic resolution by x-ray crystallography. This has revealed a common fold with amino acid-binding bacterial proteins and with the glutamate-binding module of AMPA-receptors and kainate-receptors.

Durable Change (Rojas, p. 21)

- NMDA receptors are ionic channels permeable for different kinds of molecules, like sodium, calcium, or potassium ions. These channels are blocked by a magnesium ion in such a way that the permeability for sodium and calcium is low.

- If the cell is **brought up to a certain excitation level**, the ionic channels lose the magnesium ion and become unblocked. The permeability for Ca^{2+} ions increases immediately. Through the flow of calcium ions, a chain of reactions is started which produces a **durable change** of the **threshold level** of the cell.

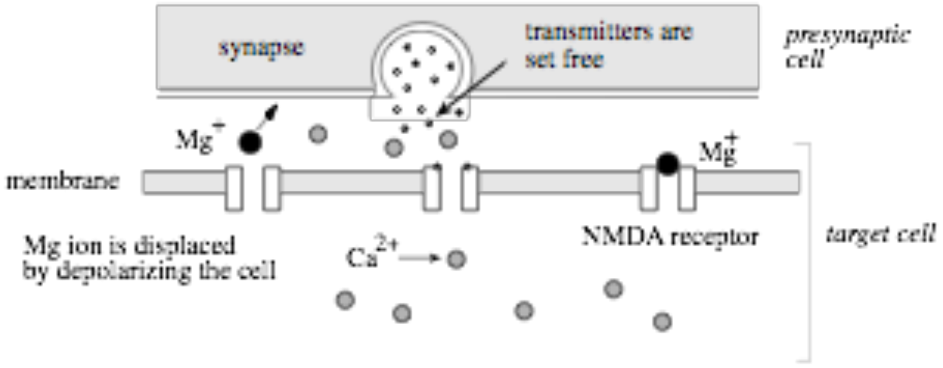
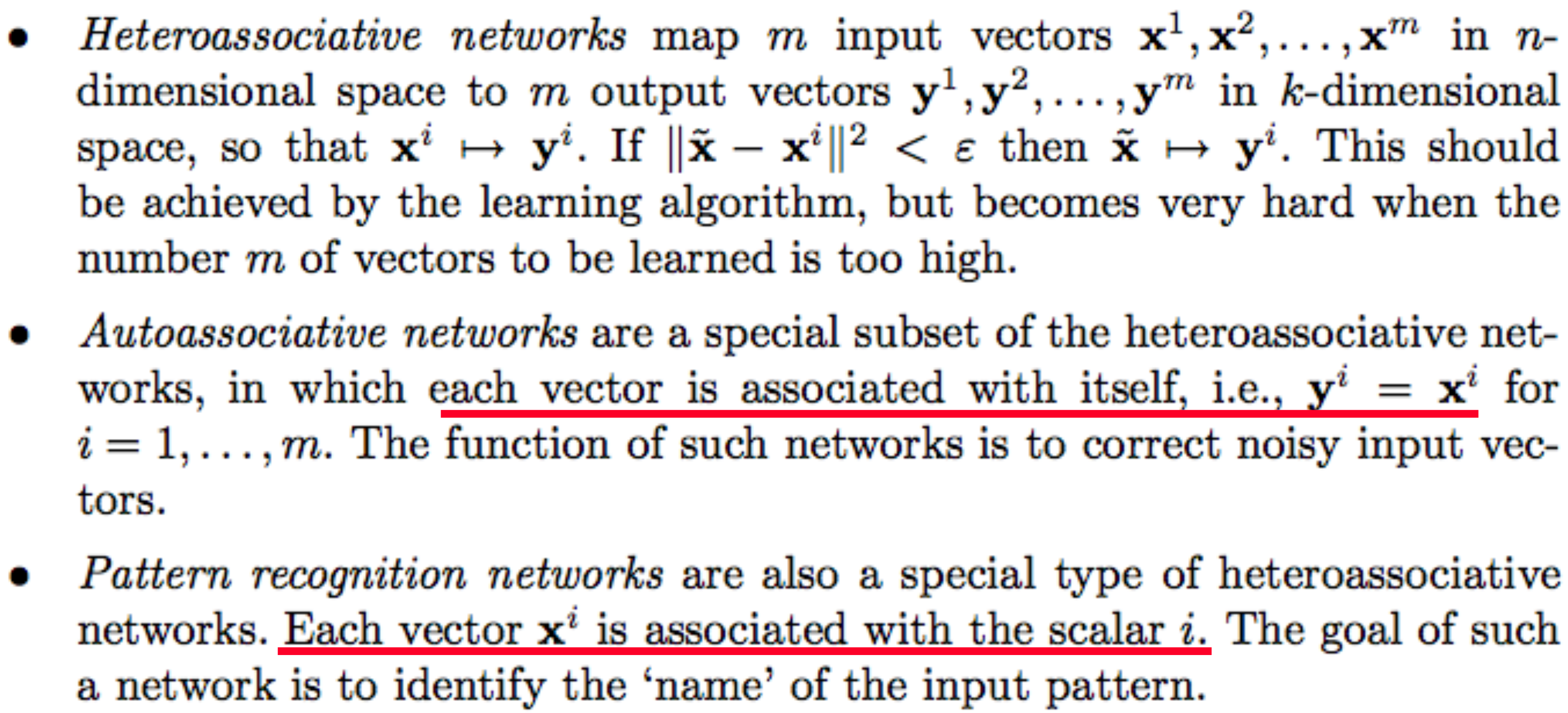


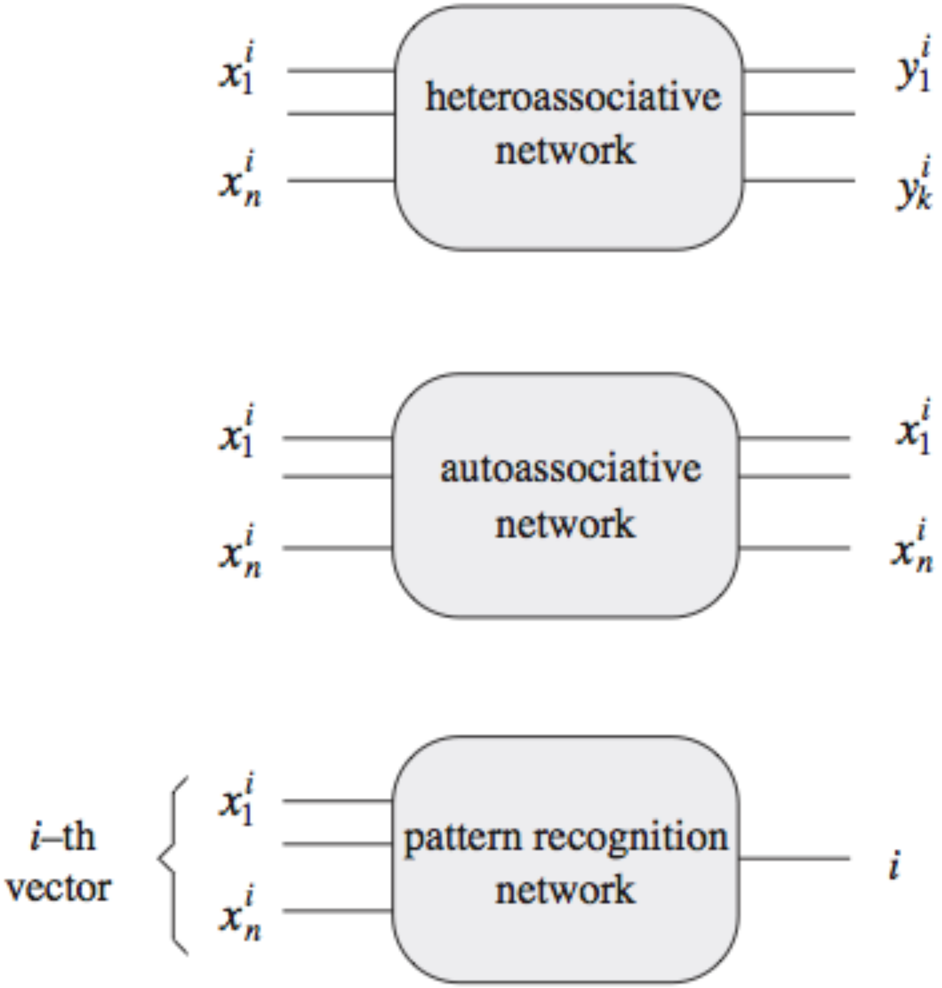
Fig. 1.13. Unblocking of an NMDA receptor

Three Kinds of Associative Networks
(Rojas, p. 312)



- *Heteroassociative networks* map m input vectors $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^m$ in n-dimensional space to m output vectors $\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^m$ in k-dimensional space, so that $\mathbf{x}^i \mapsto \mathbf{y}^i$. If $\|\tilde{\mathbf{x}} - \mathbf{x}^i\|^2 < \varepsilon$ then $\tilde{\mathbf{x}} \mapsto \mathbf{y}^i$. This should be achieved by the learning algorithm, but becomes very hard when the number m of vectors to be learned is too high.
- *Autoassociative networks* are a special subset of the heteroassociative networks, in which each vector is associated with itself, i.e., $\mathbf{y}^i = \mathbf{x}^i$ for $i = 1, \ldots, m$. The function of such networks is to correct noisy input vectors.
- *Pattern recognition networks* are also a special type of heteroassociative networks. Each vector $\mathbf{x}^i$ is associated with the scalar i. The goal of such a network is to identify the 'name' of the input pattern.

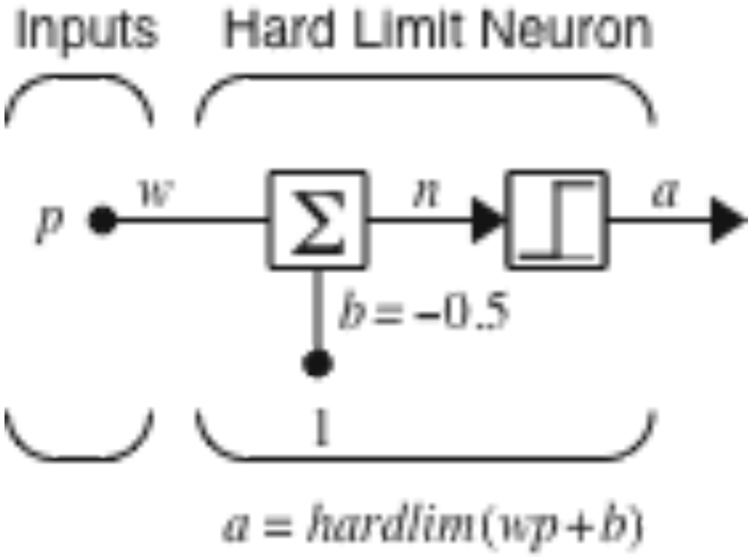
Rojas, p. 313


x_1^i
x_n^i
heteroassociative network
y_1^i
y_k^i
x_1^i
x_n^i
autoassociative network
x_1^i
x_n^i
i–th vector
x_1^i
x_n^i
pattern recognition network
i

Linear Associative Networks

- James Anderson, 1972
- Teuvo Kohonen, 1972

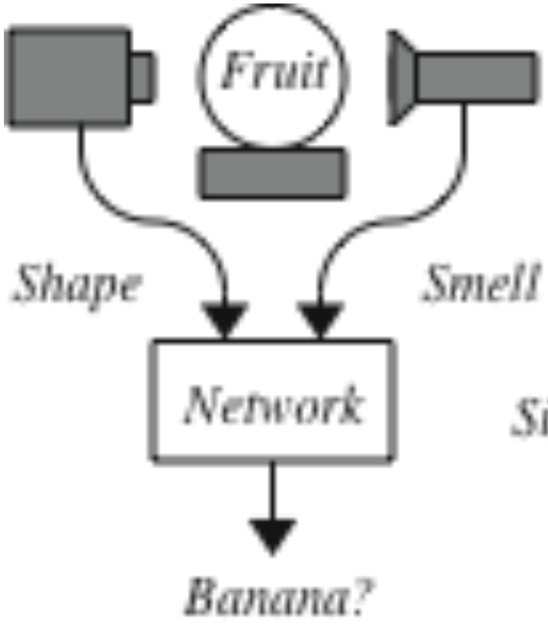
Simple Associative Network

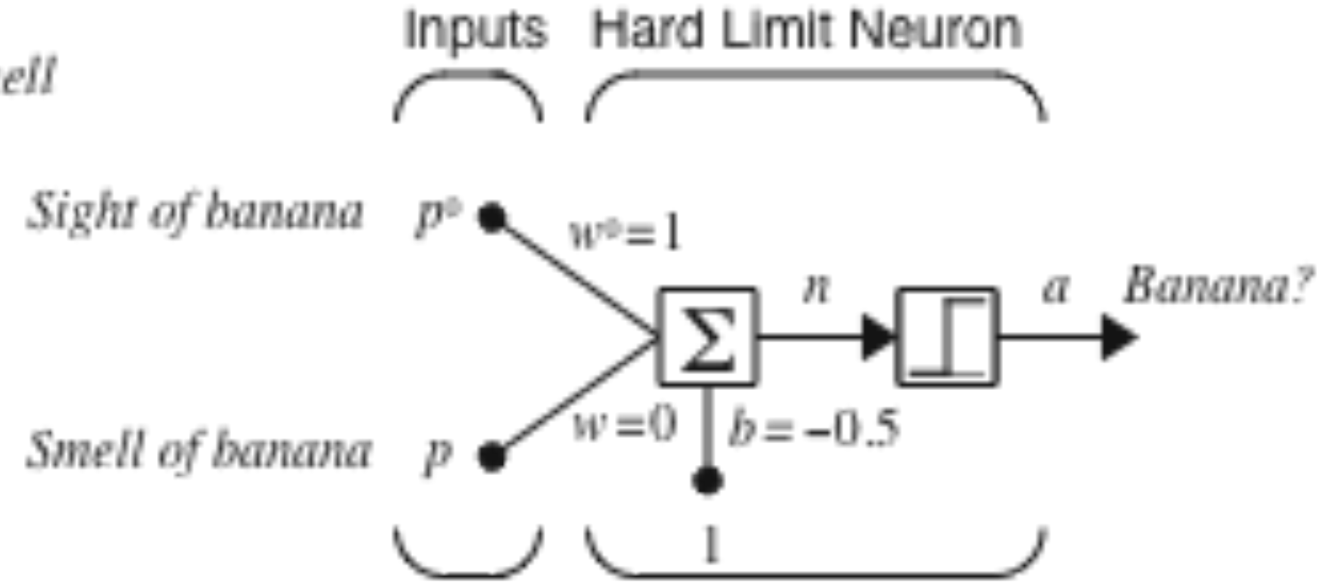


$$a = hardlim(wp + b) = hardlim(wp - 0.5)$$

$$p = \begin{cases} 1, & \text{stimulus} \\ 0, & \text{no stimulus} \end{cases} \qquad a = \begin{cases} 1, & \text{response} \\ 0, & \text{no response} \end{cases}$$

Banana Associator





$$a = hardlim(w^0 p^0 + w p + b)$$

Unconditioned Stimulus

$$p^0 = \begin{cases} 1, & \text{shape detected} \\ 0, & \text{shape not detected} \end{cases}$$

Conditioned Stimulus

$$p = \begin{cases} 1, & \text{smell detected} \\ 0, & \text{smell not detected} \end{cases}$$

Unsupervised Hebb Rule

$$w_{ij}(\mathrm{q}) = w_{ij}(q-1) + \alpha a_i(q) p_j(q)$$

a designates the output,
p the input pattern

Vector Form:

$$\boxed{\mathbf{W}(q) = \mathbf{W}(q-1) + \alpha \mathbf{a}(q) \mathbf{p}^T(q)}$$

Training Sequence:

$$\mathbf{p}(1), \mathbf{p}(2), \dots, \mathbf{p}(Q)$$

Banana Recognition Example

Initial Weights:

$$w^0 = 1, w(0) = 0$$

Training Sequence:

$$\{p^0(1) = 0, p(1) = 1\}, \{p^0(2) = 1, p(2) = 1\}, \dots$$

$$\boxed{\alpha = 1}$$

$$w(q) = w(q-1) + a(q)p(q)$$

First Iteration (sight fails):

$$\begin{aligned} a(1) &= hardlim(w^0 p^0(1) + w(0)p(1) - 0.5) \\ &= hardlim(1 \cdot 0 + 0 \cdot 1 - 0.5) = 0 \quad \text{(no response)} \end{aligned}$$

$$w(1) = w(0) + a(1)p(1) = 0 + 0 \cdot 1 = 0$$

Example

Second Iteration (sight works):

$$
\begin{aligned}
a(2) &= hardlim(w^0 p^0(2) + w(1)p(2) - 0.5) \\
&= hardlim(1 \cdot 1 + 0 \cdot 1 - 0.5) = 1 \quad \text{(banana)}
\end{aligned}
$$

$$w(2) = w(1) + a(2)p(2) = 0 + 1 \cdot 1 = 1$$

Third Iteration (sight fails):

$$
\begin{aligned}
a(3) &= hardlim(w^0 p^0(3) + w(2)p(3) - 0.5) \\
&= hardlim(1 \cdot 0 + 1 \cdot 1 - 0.5) = 1 \quad \text{(banana)}
\end{aligned}
$$

$$w(3) = w(2) + a(3)p(3) = 1 + 1 \cdot 1 = 2$$

Banana will now be detected if either sensor works.

Problems with Hebb Rule Math

- Weights can become arbitrarily large

Hebb Rule with Decay

$$\mathbf{W}(q) = \mathbf{W}(q-1) + \alpha\mathbf{a}(q)\mathbf{p}^T(q) - \gamma\mathbf{W}(q-1)$$

$$\boxed{\mathbf{W}(q) = (1-\gamma)\mathbf{W}(q-1) + \alpha\mathbf{a}(q)\mathbf{p}^T(q)}$$

This keeps the weight matrix from growing without bound, which can be demonstrated by setting both a_i and p_j to 1:

$$w_{ij}^{max} = (1-\gamma)w_{ij}^{max} + \alpha a_i p_j$$

$$w_{ij}^{max} = (1-\gamma)w_{ij}^{max} + \alpha$$

$$w_{ij}^{max} = \frac{\alpha}{\gamma}$$

Example: Banana Associator

$\alpha = 1$ $\boxed{\gamma = 0.1}$

First Iteration (sight fails):

$$\begin{aligned} a(1) &= hardlim(w^0 p^0(1) + w(0)p(1) - 0.5) \\ &= hardlim(1 \cdot 0 + 0 \cdot 1 - 0.5) = 0 \quad \text{(no response)} \end{aligned}$$

$$w(1) = w(0) + a(1)p(1) - 0.1w(0) = 0 + 0 \cdot 1 - 0.1(0) = 0$$

Second Iteration (sight works):

$$\begin{aligned} a(2) &= hardlim(w^0 p^0(2) + w(1)p(2) - 0.5) \\ &= hardlim(1 \cdot 1 + 0 \cdot 1 - 0.5) = 1 \quad \text{(banana)} \end{aligned}$$

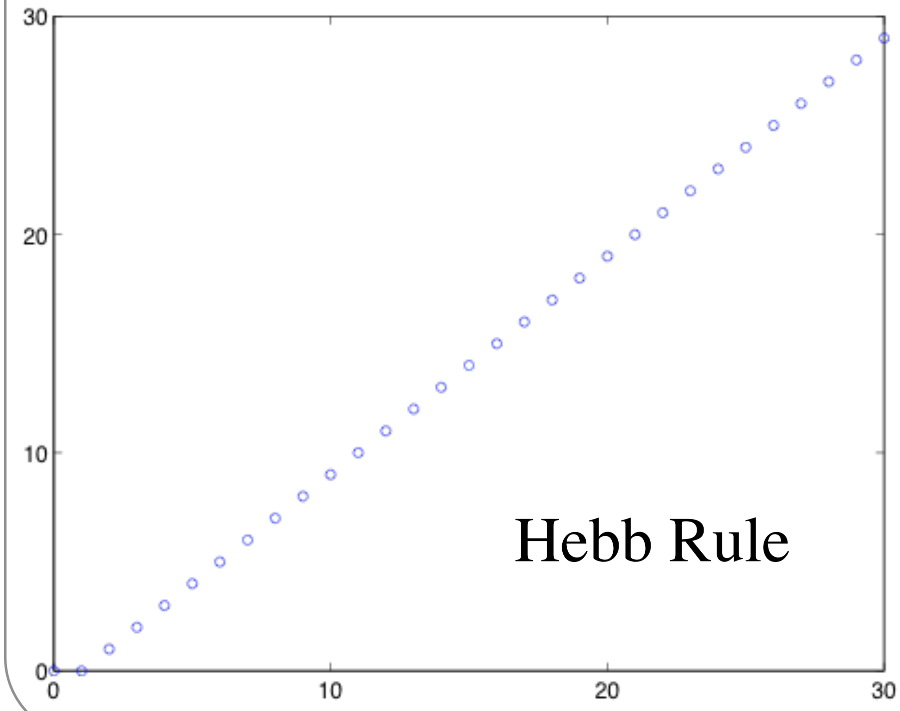
$$w(2) = w(1) + a(2)p(2) - 0.1w(1) = 0 + 1 \cdot 1 - 0.1(0) = 1$$

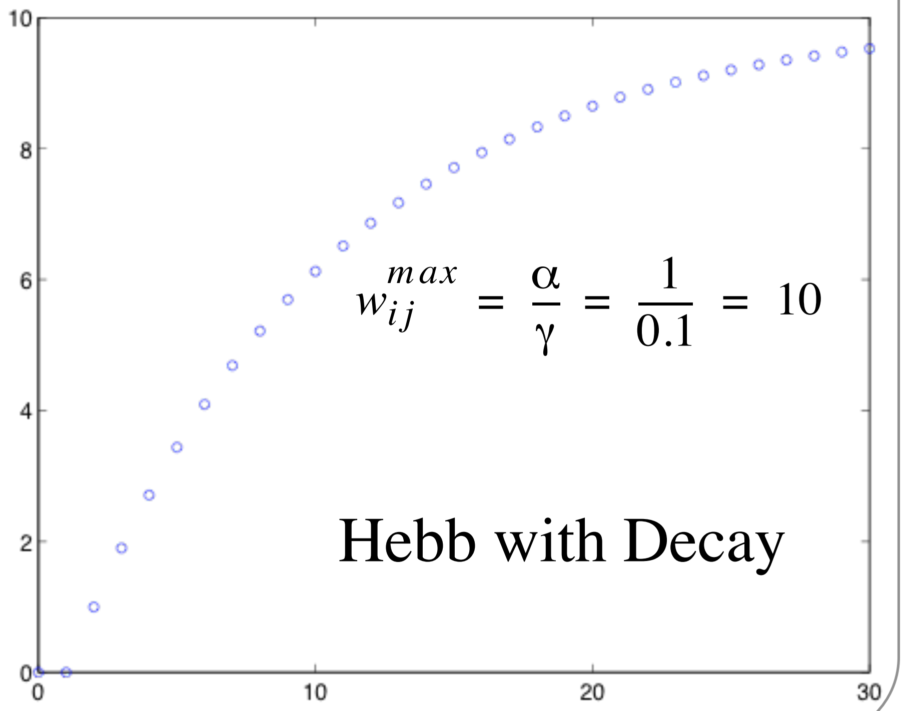
Example

Third Iteration (sight fails):

$$
\begin{aligned}
a(3) &= hardlim\,(w^0 p^0(3) + w(2)\,p(3) - 0.5) \\
&= hardlim\,(1 \cdot 0 + 1 \cdot 1 - 0.5) = 1 \quad \text{(banana)}
\end{aligned}
$$

$$w(3) = w(2) + a(3)p(3) - 0.1w(3) = 1 + 1 \cdot 1 - 0.1(1) = 1.9$$





$$w_{ij}^{max} = \frac{\alpha}{\gamma} = \frac{1}{0.1} = 10$$

Problem of Hebb with Decay

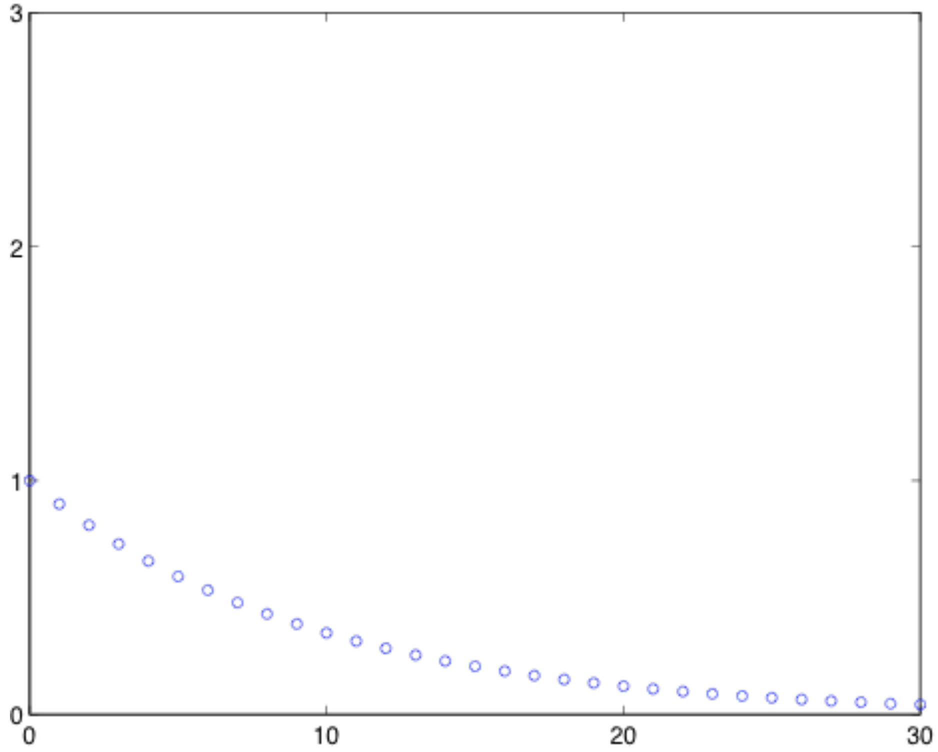
Associations will decay away if stimuli are not occasionally re-presented.

If $a_i = 0$, then

$$w_{ij}(q) = (1 - \gamma)w_{ij}(q-1)$$

If $\gamma = 0$, this becomes

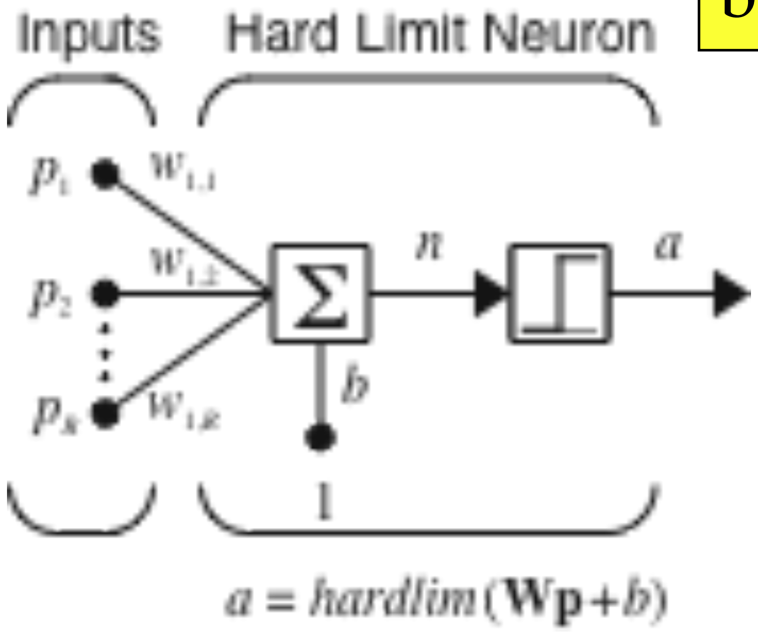
$$w_{ij}(q) = (0.9)w_{ij}(q-1)$$



Therefore the weight decays by 10% at each iteration where there is no stimulus.

“Instar” (Recognition Network), Grossberg
Purpose is to learn a single pattern

Appears to be
a perceptron
by another name.



Instar Operation

$$a = hardlim(\mathbf{W}\mathbf{p} + b) = hardlim({}_1\mathbf{w}^T\mathbf{p} + b)$$

The instar will be active when

$${}_1\mathbf{w}^T\mathbf{p} \geq -b$$

or

$${}_1\mathbf{w}^T\mathbf{p} = \|{}_1\mathbf{w}\|\|\mathbf{p}\|\cos\theta \geq -b$$

For normalized vectors, the largest inner product occurs when the angle between the weight vector and the input vector is zero -- the input vector is equal to the weight vector.

The rows of a weight matrix represent patterns to be recognized.

Vector Recognition by Instar

If we set

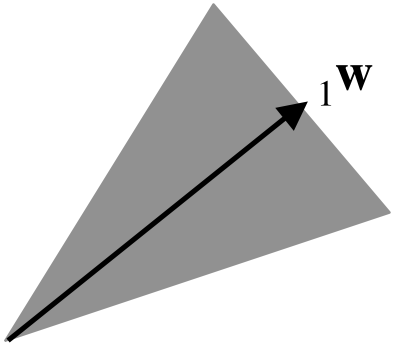
$$b = -\|{}_1\mathbf{w}\|\|\mathbf{p}\|$$

the instar will only be active when $\theta = 0$.

If we set

$$b > -\|{}_1\mathbf{w}\|\|\mathbf{p}\|$$

the instar will be active for a range of angles.



As *b* is increased, the more patterns there will be (over a wider range of θ) which will activate the instar.

Instar Rule

Hebb with Decay

$$w_{ij}(q) = w_{ij}(q-1) + \alpha a_i(q) p_j(q)$$

Modify so that learning and forgetting will only occur when the neuron is active - Instar Rule:

$$w_{ij}(q) = w_{ij}(q-1) + \alpha a_i(q) p_j(q) \underbrace{- \gamma a_i(q) w_{ij}(q-1)}_{\text{“forgetting” term}}$$

or

$$w_{ij}(q) = w_{ij}(q-1) + \alpha a_i(q)\left(p_j(q) - w_{ij}(q-1)\right)$$

(*if* we make the decay rate γ equal to the learning rate α)

Vector Form:

$$\boxed{{}_i\mathbf{w}(q) = {}_i\mathbf{w}(q-1) + \alpha a_i(q)\left(\mathbf{p}(q) - {}_i\mathbf{w}(q-1)\right)}$$

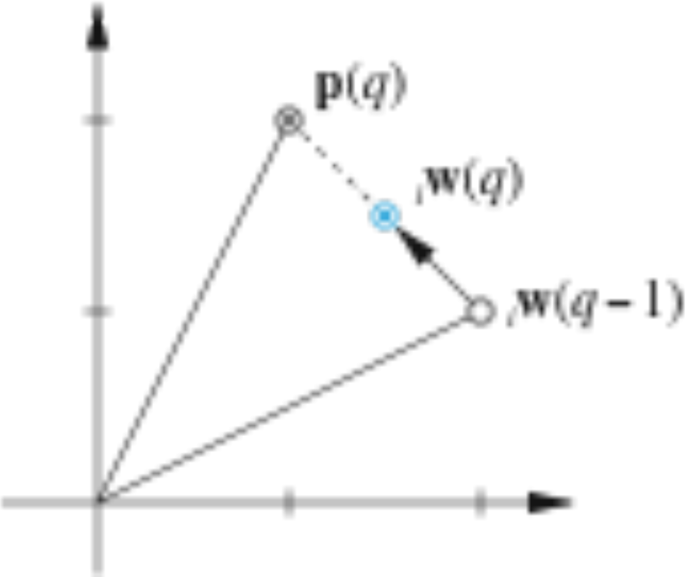
Graphical Representation

For the case where the instar is active ($a_i = 1$):

$${}_i\mathbf{w}(q) = {}_i\mathbf{w}(q-1) + \alpha(\mathbf{p}(q) - {}_i\mathbf{w}(q-1))$$

or

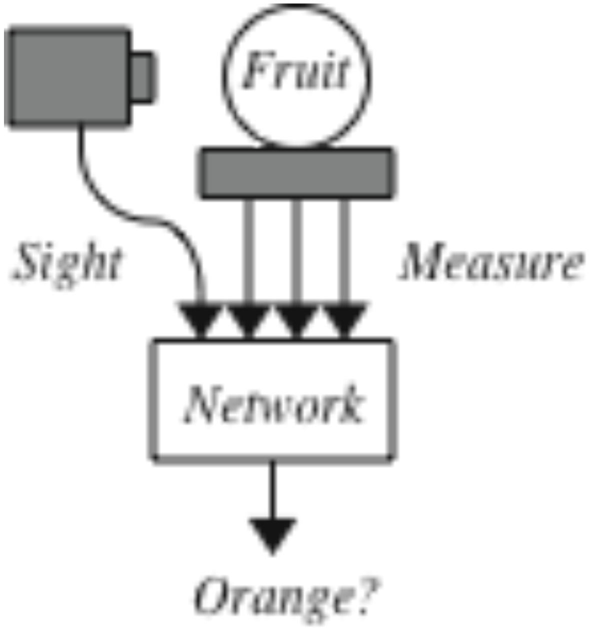
$${}_i\mathbf{w}(q) = (1-\alpha)\,{}_i\mathbf{w}(q-1) + \alpha\mathbf{p}(q)$$



For the case where the instar is inactive ($a_i = 0$):

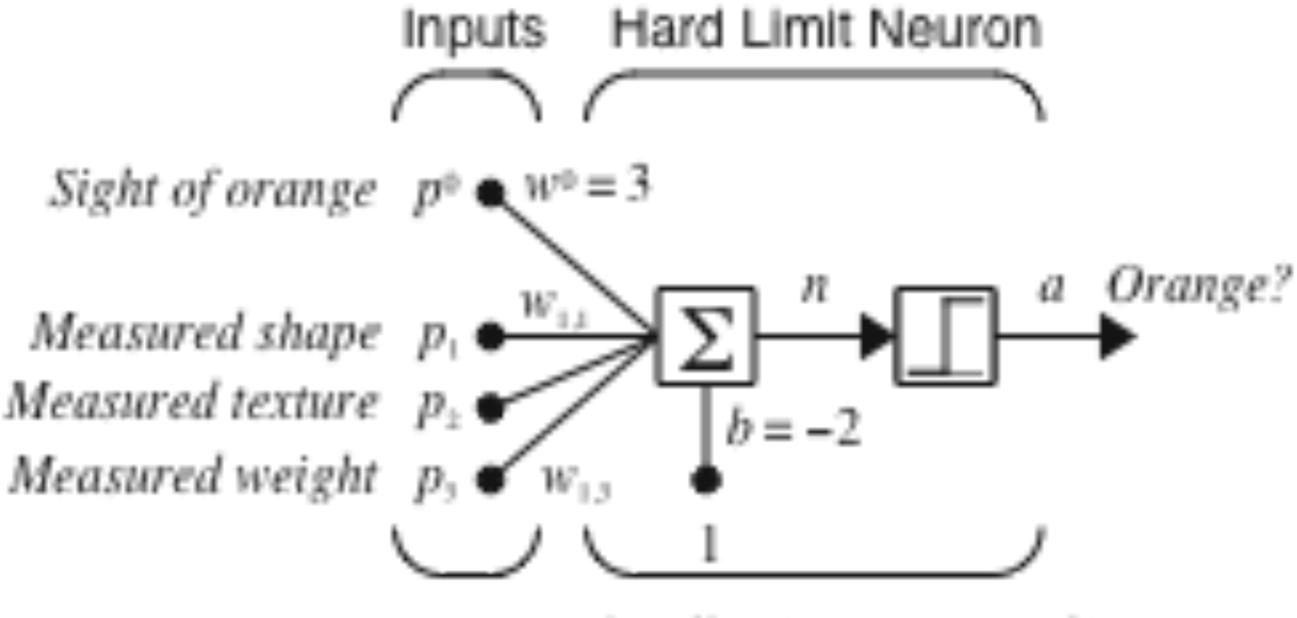
$${}_i\mathbf{w}(q) = {}_i\mathbf{w}(q-1)$$

Example



$$p^0 = \begin{cases} 1, & \text{orange detected visually} \\ 0, & \text{orange not detected} \end{cases}$$

$$\mathbf{p} = \begin{bmatrix} shape \\ texture \\ weight \end{bmatrix}$$



$$a = hardlim(w^0 p^0 + \mathbf{W}\mathbf{p} + b)$$

Training Instar

$$\mathbf{W}(0) = {}_1\mathbf{w}^T(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ p^0(1) = 0, \mathbf{p}(1) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}, \left\{ p^0(2) = 1, \mathbf{p}(2) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}, \ldots$$

First Iteration (α=1):

$$a(1) = hardlim(w^0 p^0(1) + \mathbf{Wp}(1) - 2)$$

$$a(1) = hardlim\left(3 \cdot 0 + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - 2 \right) = 0 \quad \text{(no response)}$$

$${}_1\mathbf{w}(1) = {}_1\mathbf{w}(0) + a(1)(\mathbf{p}(1) - {}_1\mathbf{w}(0)) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0 \left(\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Further Training of Instar

$$a(2) = hardlim(w^0 p^0(2) + \mathbf{W}\mathbf{p}(2) - 2) = hardlim\left(3 \cdot 1 + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - 2\right) = 1$$

(orange)

$${}_1\mathbf{w}(2) = {}_1\mathbf{w}(1) + a(2)(\mathbf{p}(2) - {}_1\mathbf{w}(1)) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 1\left(\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$a(3) = hardlim(w^0 p^0(3) + \mathbf{W}\mathbf{p}(3) - 2) = hardlim\left(3 \cdot 0 + \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - 2\right) = 1$$

(orange)

$${}_1\mathbf{w}(3) = {}_1\mathbf{w}(2) + a(3)(\mathbf{p}(3) - {}_1\mathbf{w}(2)) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + 1\left(\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Orange will now be detected if *either* set of sensors works.

Kohonen Rule

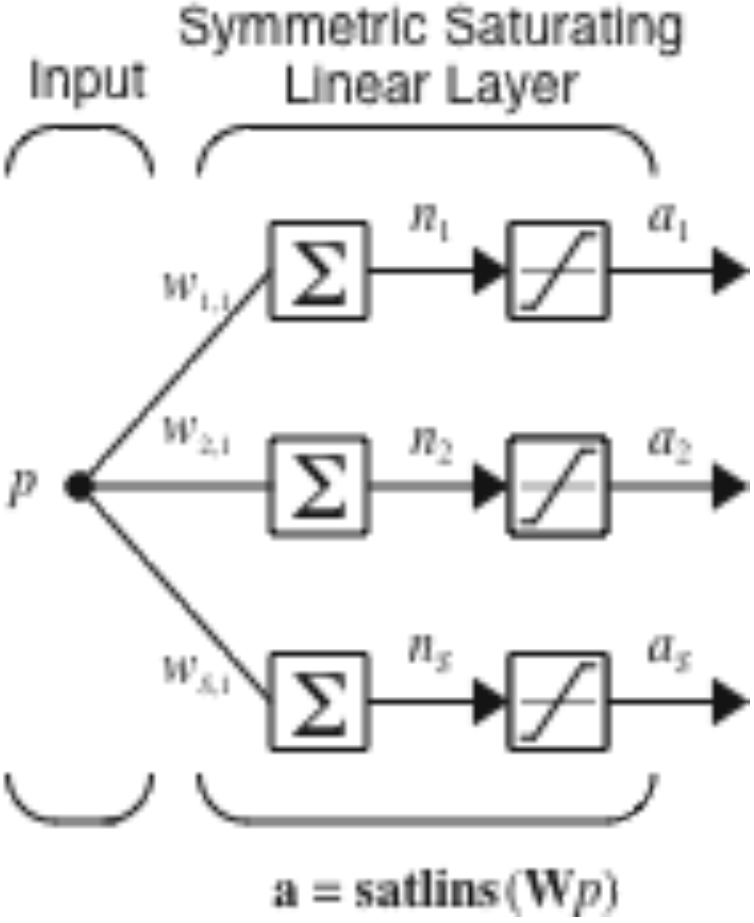
$${}_1\mathbf{w}(q) = {}_1\mathbf{w}(q-1) + \alpha(\mathbf{p}(q) - {}_1\mathbf{w}(q-1)), \quad \boxed{\text{for } i \in X(q)}$$

Learning occurs when the neuron’s index *i* is a member of the “neighborhood” set $X(q)$.

All neurons in a given neighborhood learn.

(Kohonen = Instar when neighborhood size = 1.)

Outstar (Recall Network) Grossberg
The “dual” of an Instar



Outstar Operation

Suppose we want the outstar to recall a certain pattern $\mathbf{a}^*$ whenever the input $p = 1$ is presented to the network. Let

$$\mathbf{W} = \mathbf{a}^*$$

Then, when $p = 1$

$$\mathbf{a} = \mathbf{satlins}(\mathbf{W}p) = \mathbf{satlins}(\mathbf{a}^* \cdot 1) = \mathbf{a}^*$$

and the pattern is correctly recalled.

The columns of a weight matrix represent patterns to be recalled.

Outstar Rule

For the instar rule we made the weight decay term of the Hebb rule proportional to the **output** of the network.
For the outstar rule we make the weight decay term proportional to the **input** of the network.

$$w_{ij}(q) = w_{ij}(q-1) + \alpha a_i(q) p_j(q) - \gamma p_j(q) w_{ij}(q-1)$$

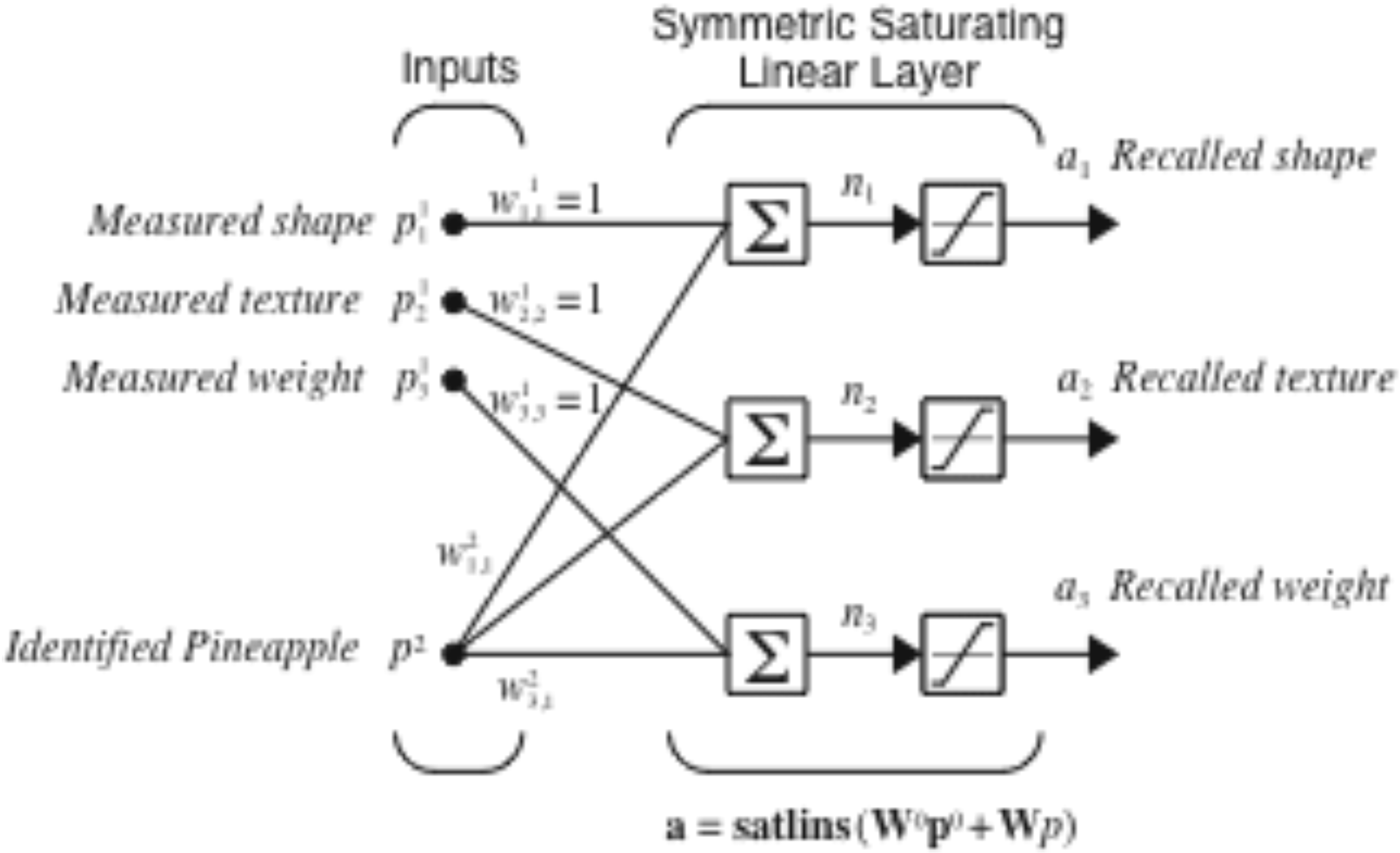
If we make the decay rate γ equal to the learning rate α,

$$w_{ij}(\mathrm{q}) = w_{ij}(q-1) + \alpha (a_i(q) - w_{ij}(q-1)) p_j(q)$$

Vector Form:

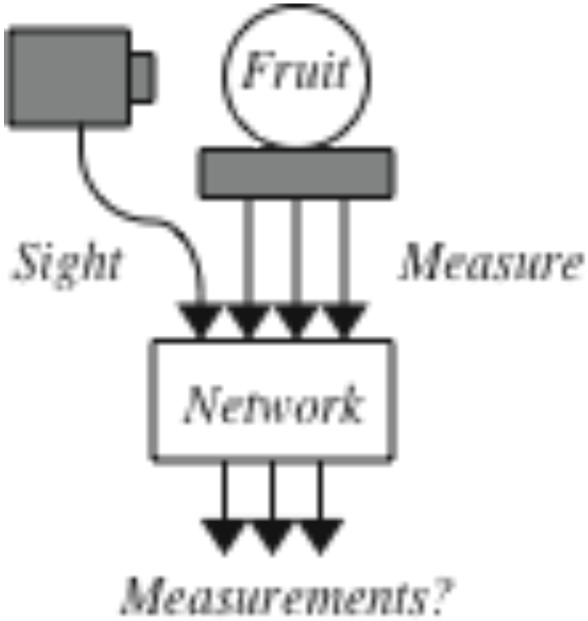
$$\boxed{\mathbf{w}_j(q) = \mathbf{w}_j(q-1) + \alpha(\mathbf{a}(q) - \mathbf{w}_j(q-1)) p_j(q)}$$

Example - Pineapple Recall



Definitions

$$\mathbf{a} = \mathbf{satlins}(\mathbf{W}^0\mathbf{p}^0 + \mathbf{W}p)$$



$$\mathbf{W}^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{p}^0 = \begin{bmatrix} shape \\ texture \\ weight \end{bmatrix} \qquad \mathbf{p}^{pineapple} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$p = \begin{cases} 1, & \text{if a pineapple can be seen} \\ 0, & \text{otherwise} \end{cases}$$

Iteration 1

$$\left\{\mathbf{p}^0(1) = \begin{bmatrix}0\\0\\0\end{bmatrix}, p(1) = 1\right\}, \left\{\mathbf{p}^0(2) = \begin{bmatrix}-1\\-1\\1\end{bmatrix}, p(2) = 1\right\}, \ldots$$

$$\alpha = 1$$

$$\mathbf{a}(1) = \mathbf{satlins}\left(\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix}1\right) = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{(no response)}$$

$$\mathbf{w}_1(1) = \mathbf{w}_1(0) + (\mathbf{a}(1) - \mathbf{w}_1(0))\,p(1) = \begin{bmatrix}0\\0\\0\end{bmatrix} + \left(\begin{bmatrix}0\\0\\0\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix}\right)1 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Convergence

$$\mathbf{a}(2) = \mathbf{satlins}\left(\begin{bmatrix}-1\\-1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix}1\right) = \begin{bmatrix}-1\\-1\\1\end{bmatrix} \quad \text{(measurements given)}$$

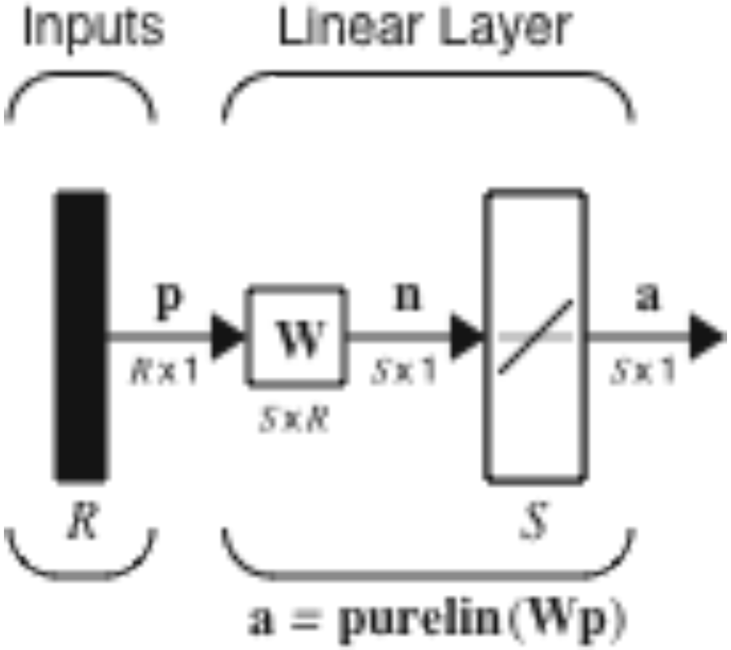
$$\mathbf{w}_1(2) = \mathbf{w}_1(1) + (\mathbf{a}(2) - \mathbf{w}_1(1))p(2) = \begin{bmatrix}0\\0\\0\end{bmatrix} + \left(\begin{bmatrix}-1\\-1\\1\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix}\right)1 = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

$$\mathbf{a}(3) = \mathbf{satlins}\left(\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}-1\\-1\\1\end{bmatrix}1\right) = \begin{bmatrix}-1\\-1\\1\end{bmatrix} \quad \text{(measurements recalled)}$$

$$\mathbf{w}_1(3) = \mathbf{w}_1(2) + (\mathbf{a}(2) - \mathbf{w}_1(2))p(2) = \begin{bmatrix}-1\\-1\\1\end{bmatrix} + \left(\begin{bmatrix}-1\\-1\\1\end{bmatrix} - \begin{bmatrix}-1\\-1\\1\end{bmatrix}\right)1 = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

Associative Learning: Supervised Hebbian Learning

Linear Associator



$$\mathbf{a} = \mathbf{W}\mathbf{p} \qquad a_i = \sum_{j=1}^{R} w_{ij} p_j$$

Training Set:

$$\{\mathbf{p}_1, \mathbf{t}_1\}, \{\mathbf{p}_2, \mathbf{t}_2\}, \ldots, \{\mathbf{p}_Q, \mathbf{t}_Q\}$$

Supervised Hebb Rule

$$w_{ij}^{new} = w_{ij}^{old} + \alpha f_i(a_{iq}) g_j(p_{jq})$$

a designates the output,
p the input pattern

Presynaptic Signal

Postsynaptic Signal

Simplified Form:

$$w_{ij}^{new} = w_{ij}^{old} + \alpha a_{iq} p_{jq}$$

Supervised Form: Replace output signal with desired target.

$$w_{ij}^{new} = w_{ij}^{old} + t_{iq} p_{jq}$$

Matrix Form:

$$\mathbf{W}^{new} = \mathbf{W}^{old} + \mathbf{t}_q \mathbf{p}_q^T$$

Batch Operation

$$\mathbf{W} = \mathbf{t}_1\mathbf{p}_1^T + \mathbf{t}_2\mathbf{p}_2^T + \cdots + \mathbf{t}_Q\mathbf{p}_Q^T = \sum_{q=1}^{Q} \mathbf{t}_q\mathbf{p}_q^T$$

(Zero Initial
Weights)

Matrix Form:

$$\mathbf{W} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \dots & \mathbf{t}_Q \end{bmatrix} \begin{bmatrix} \mathbf{p}_1^T \\ \mathbf{p}_2^T \\ \vdots \\ \mathbf{p}_Q^T \end{bmatrix} = \mathbf{T}\mathbf{P}^T$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_Q \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \dots & \mathbf{t}_Q \end{bmatrix}$$

(Generally, the targets would be column vectors, and we have *outer* products between the targets and the patterns.)

Performance Analysis

$$\mathbf{a} = \mathbf{W}\mathbf{p}_k = \left(\sum_{q=1}^{Q} \mathbf{t}_q \mathbf{p}_q^T\right)\mathbf{p}_k = \sum_{q=1}^{Q} \mathbf{t}_q (\mathbf{p}_q^T \mathbf{p}_k)$$

Case I, input patterns are orthogonal.

$$\begin{aligned} (\mathbf{p}_q^T \mathbf{p}_k) &= 1 \qquad q = k \\ &= 0 \qquad q \neq k \end{aligned}$$

Therefore the network output equals the target:

$$\mathbf{a} = \mathbf{W}\mathbf{p}_k = \mathbf{t}_k$$

Case II, input patterns are normalized, but not orthogonal.

$$\mathbf{a} = \mathbf{W}\mathbf{p}_k = \mathbf{t}_k + \boxed{\sum_{q \neq k} \mathbf{t}_q (\mathbf{p}_q^T \mathbf{p}_k)}$$

Error term

Example

Banana Apple Normalized Prototype Patterns

$$\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad \left\{ \mathbf{p}_1 = \begin{bmatrix} -0.5774 \\ 0.5774 \\ -0.5774 \end{bmatrix}, \mathbf{t}_1 = \begin{bmatrix} -1 \end{bmatrix} \right\} \quad \left\{ \mathbf{p}_2 = \begin{bmatrix} 0.5774 \\ 0.5774 \\ -0.5774 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 1 \end{bmatrix} \right\}$$

Weight Matrix (Hebb Rule):

$$\mathbf{W} = \mathbf{T}\mathbf{P}^T = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -0.5774 & 0.5774 & -0.5774 \\ 0.5774 & 0.5774 & -0.5774 \end{bmatrix} = \begin{bmatrix} 1.1548 & 0 & 0 \end{bmatrix}$$

Tests:

Banana $$\mathbf{W}\mathbf{p}_1 = \begin{bmatrix} 1.1548 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.5774 \\ 0.5774 \\ -0.5774 \end{bmatrix} = \begin{bmatrix} -0.6668 \end{bmatrix}$$

Apple $$\mathbf{W}\mathbf{p}_2 = \begin{bmatrix} 0 & 1.1548 & 0 \end{bmatrix} \begin{bmatrix} 0.5774 \\ 0.5774 \\ -0.5774 \end{bmatrix} = \begin{bmatrix} 0.6668 \end{bmatrix}$$

Pseudoinverse Rule - (1)

Performance Index: $\mathbf{W}\mathbf{p}_q = \mathbf{t}_q \qquad q = 1, 2, \ldots, Q$

$$F(\mathbf{W}) = \sum_{q=1}^{Q} \| \mathbf{t}_q - \mathbf{W}\mathbf{p}_q \|^2$$

Matrix Form:

$$\mathbf{W}\mathbf{P} = \mathbf{T}$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \cdots & \mathbf{t}_Q \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_Q \end{bmatrix}$$

$$F(\mathbf{W}) = \| \mathbf{T} - \mathbf{W}\mathbf{P} \|^2 = \| \mathbf{E} \|^2$$

$$\| \mathbf{E} \|^2 = \sum_i \sum_j e_{ij}^2$$

Pseudoinverse Rule - (2)

$$\mathbf{W}\mathbf{P} = \mathbf{T}$$

Minimize:

$$F(\mathbf{W}) = \|\mathbf{T} - \mathbf{W}\mathbf{P}\|^2 = \|\mathbf{E}\|^2$$

If an inverse exists for **P**, $F(\mathbf{W})$ can be made zero:

$$\mathbf{W} = \mathbf{T}\mathbf{P}^{-1}$$

When an inverse does not exist $F(\mathbf{W})$ can be minimized using the pseudoinverse:

$$\mathbf{W} = \mathbf{T}\mathbf{P}^{+}$$

$$\mathbf{P}^{+} = (\mathbf{P}^{T}\mathbf{P})^{-1}\mathbf{P}^{T}$$

Relationship to the Hebb Rule

Hebb Rule

$$\mathbf{W} = \mathbf{T}\mathbf{P}^T$$

Pseudoinverse Rule

$$\mathbf{W} = \mathbf{T}\mathbf{P}^+$$

$$\mathbf{P}^+ = (\mathbf{P}^T\mathbf{P})^{-1}\mathbf{P}^T$$

If the prototype patterns are orthonormal the two rules are equivalent:

$$\mathbf{P}^T\mathbf{P} = \mathbf{I}$$

$$\mathbf{P}^+ = (\mathbf{P}^T\mathbf{P})^{-1}\mathbf{P}^T = \mathbf{P}^T$$

Example

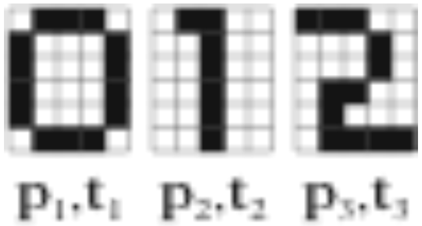
$$\left\{\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{t}_1 = \begin{bmatrix} -1 \end{bmatrix}\right\} \quad \left\{\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 1 \end{bmatrix}\right\} \qquad \mathbf{W} = \mathbf{T}\mathbf{P}^{+} = \begin{bmatrix} -1 & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}\right)^{+}$$

$$\mathbf{P}^{+} = (\mathbf{P}^T\mathbf{P})^{-1}\mathbf{P}^T = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.25 & -0.25 \\ 0.5 & 0.25 & -0.25 \end{bmatrix}$$

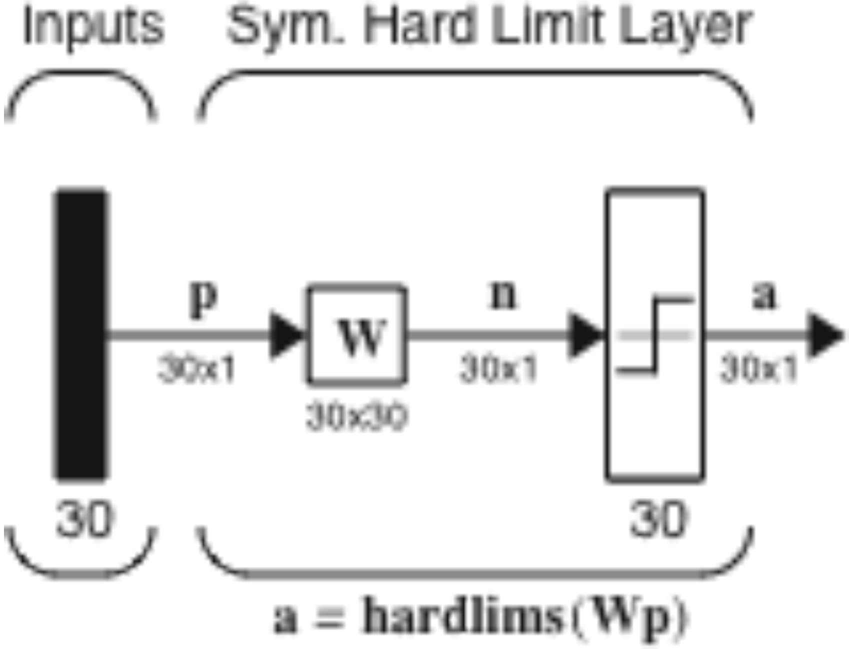
$$\mathbf{W} = \mathbf{T}\mathbf{P}^{+} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -0.5 & 0.25 & -0.25 \\ 0.5 & 0.25 & -0.25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{W}\mathbf{p}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \qquad \mathbf{W}\mathbf{p}_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

Autoassociative Memory Demo



$$\mathbf{p}_1 = \begin{bmatrix} -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & \ldots & 1 & -1 \end{bmatrix}^T$$



$$\mathbf{W} = \mathbf{p}_1\mathbf{p}_1^T + \mathbf{p}_2\mathbf{p}_2^T + \mathbf{p}_3\mathbf{p}_3^T$$

(Hebb Rule)

Tests

50% Occluded Patterns

67% Occluded Patterns

Noisy Patterns (7 pixels)

Variations of Hebbian Learning

Basic Rule: $\mathbf{W}^{new} = \mathbf{W}^{old} + \mathbf{t}_q \mathbf{p}_q^T$

Learning Rate: $\mathbf{W}^{new} = \mathbf{W}^{old} + \alpha\, \mathbf{t}_q \mathbf{p}_q^T$

Smoothing: $\mathbf{W}^{new} = \mathbf{W}^{old} + \alpha\, \mathbf{t}_q \mathbf{p}_q^T - \gamma \mathbf{W}^{old} = (1 - \gamma)\mathbf{W}^{old} + \alpha \mathbf{t}_q \mathbf{p}_q^T$

Delta Rule: $\mathbf{W}^{new} = \mathbf{W}^{old} + \alpha(\mathbf{t}_q - \mathbf{a}_q)\mathbf{p}_q^T$

Unsupervised: $\mathbf{W}^{new} = \mathbf{W}^{old} + \alpha \mathbf{a}_q \mathbf{p}_q^T$

Related Topics to be Studied

- Hopfield Networks
- PCA Networks
  - Oja's rule, and others