Low-Level Functional Programming

Bob Keller

September 2010

What's "Low-Level" About This?

- "low-level" refers to the construction of functions by explicitly creating and decomposing lists **a few elements at a time**.
- Previously we used higher-order functions to do most of the non-trivial work in a functional decomposition.
- Now we are going to use **recursion**.

Fundamental List Dichotomy

- A list is either:
  - **empty**, () or
  - **non-empty**, in which case it has both a
    - **first**
    - **rest**
- Most list function definitions deal with these two cases separately.
- Definitions are typically a form of **inductive definition**, in which empty is the basis.

Defining Functions on Lists

- Suppose we want to define a function taking an arbitrary list as an argument.
- It is sufficient to:
  - define the function on the empty list, and
  - define the function on a general non-empty list.

called the "basis"

called the "induction step" or "recursion"

Example

- Define the function **halve_all,** which divides every element in a list by 2 (of course this could be done using map, but this is for illustration):
  - ```
    (define (halve_all L)
      (if (null? L)
        '()
        (cons (halve (first L)) (halve_all (rest L)))))
    ```
- This can be read:
  - "halving all of the empty list is the empty list."
  - "halving all of a non-empty list is half of the first element ***followed by*** halving all of the rest."

Computation by "Rewriting"

- (halve_all '(2 4 6)) ⇒
- (cons 1 (halve_all '(4 6)) ⇒
- (cons 1 (cons 2 (halve_all '(6)))) ⇒
- (cons 1 (cons 2 (cons 3 (halve_all '())))) ⇒
- (cons 1 (cons 2 (cons 3 '()))) ⇒
- (cons 1 (cons 2 '(3))) ⇒
- (cons 1 '(2 3)) ⇒
- '(1 2 3)
    ```

Alternate

- Of course, we could have just used ***map*** *in this particular case*:
  - (define (halve A) (/ A 2))
  - (define (halve_all X) (map halve X))
- or just
  - (define (halve_all X) (map (lambda(Y) (/ Y 2)) X))
- Use higher order functions such as *map* when possible; resort to lower-order ones when you think you need to.
- Higher-order functions can often tell the story more succinctly.

Define from a low-level:

- map
- member
- filter
- foldr
- foldl
- range: generates a list (M … N)

Example: map

```
(define (map F L)
  (if (null? L)
      '()
      (cons (F (first L))
            (map F (rest L)))))
```

Example

- Define the function **member** which tests whether the first argument is an element of the list in the second argument.
  If it is, then the entire list suffix beginning with that element is returned. Otherwise #f is returned.

```
(define (member X L)
        (if (null? L)
            #f
            (if (equal? X (first L))
               L
              (member X (rest L)))))
```

Alternate Version: Use cond

- Better readability than with nested if's:

```
(define (member X L)
        (cond
          ((null? L)             #f)
          ((equal? X (first L))  L)
          (else                  (member X (rest L)))))

        ;;  (test condition       result)
```

range
(range 1 5) = '(1 2 3 4 5)

```
(define (range m n)
  (if (> m n)
      '()
      (cons m (range (+ m 1) n))))
```

Matching with Two or More List Arguments

- Some functions have more than one list argument.
- Induction might, or might not, use rules that dichotomize **both** lists.

Example: append

```
(define (append L M)
 (if (null? L) M
     (cons (first L)
           (append (rest L) M))))
```

The Merge Pattern

- This **important** pattern arises many times in different applications.
- Construct a function that merges two lists of numbers:
  - The argument lists are already in ascending order.
  - The result is to be in ascending order as well.

Merge Example

```
(check-expect (merge '(3 5 8 10) '(2 6 7 9 10 11)) '(2 3 5 6 7 8 9 10 10

(define (merge L M)
 (cond
  ((null? L)              M)
  ((null? M)              L)
  ((<= (first L) (first M)) (cons (first L) (merge (rest L) M)))
  (else                   (cons (first M) (merge L (rest M))))))

; Merge retains duplicates
```

Using Merge for Sorting

```
(define (mergesort L)
 (cond
  ((null? L)         '())
  ((null? (rest L)) L)
  (else              (merge (mergesort (everyother L))
                            (mergesort (everyother (rest L)))))))

(check-expect (mergesort '(5 1 9 3 2 6 4 8 7 11 10)) '(1 2 3 4 5 6 7 8 9 1
```

Exercise: Define everyother.

Using Merge Pattern for Sets

- Assume we want to build a library for manipulating sets.
- Assume that the elements are drawn from a set for which there is an ordering relation (such as numbers).

Sets as Lists

- For mathematical sets:
  - Duplicate elements do not matter
  - Order does not matter
- For lists, the above are not true.
  - To represent sets as lists, we will agree:
    - A representation will not have duplicates.
    - The elements of a representation will always be in order.
- These conventions allow efficiency gains by using the merge technique.

Operations on Two Sets

- **union**: The elements that are in either of the sets.
- **intersection**: The elements that are in both.
- **difference**: The elements that are in the first, but not the second.

union

```
(define (union A B)
  (cond
    ((null? A)                  B)
    ((null? B)                  A)
    ((= (first A) (first B)) (cons (first A) (union (rest A) (rest B))))
    ((< (first A) (first B)) (cons (first A) (union (rest A) B)))
    (else                       (cons (first B) (union A (rest B))))))

(check-expect (union '(1 2 4 6) '(3 4 5 6 7)) '(1 2 3 4 5 6 7))
```

intersection

```
(define (intersection A B)
  (cond
    ((null? A)                  '())
    ((null? B)                  '())
    ((= (first A) (first B)) (cons (first A) (intersection (rest A) (rest B))))
    ((< (first A) (first B)) (intersection (rest A) B))
    (else                       (intersection A (rest B)))))

(check-expect (intersection '(1 2 4 6) '(3 4 5 6 7)) '(4 6))
```

difference

```
(define (difference A B)



(check-expect (difference '(1 2 4 6) '(3 4 5 6 7)) '(1 2))
```

Similar Example

- Consider the representation of numbers as lists of prime factors with multiplicity, e.g.
  400 represented as ((2 4) (5 2))
  with factors in **increasing** order.
- Want to define functions gcd and lcm on this representation:
  - (gcd R S) representation of greatest common divisor of R and S
  - (lcm R S) representation of least common multiple of R and S
- (check-expect (gcd '((3 5) (5 2) (7 4)) '((3 2) (7 3) (11 2))) '((3 2) (7 3)))
  (check-expect (lcm '((3 5) (5 2) (7 4)) '((3 2) (7 3) (11 2))) '((3 5) (5 2) (7 4) (11 2)))

Unicalc Example

- Representing “quantities” in 3 parts:
  - numeric multiplier
  - symbolic numerator
  - symbolic denominator
- e.g.
  - 2/3
  - ‘(kg meter)
  - ‘(second second)

$2/3$ kg meter/sec^2

Normalized Quantities

- Units are sorted
- No units common to numerator and denominator (could be cancelled otherwise)
- Only “basic” units: Non-basic units must first be converted to basic ones using the database and recursion.

Basic Unit

- Any unit having a definition in the database association list is *not* basic.
- These are **not** basic, as they have a definition in the database:

```
(inch      (0.02539954113 (meter)()))
(hour      (60 (minute)()))
(minute    (60 (second)()))
```

- These are **basic**, as they have no definition in the database: :

```
meter
kg
second
```

Loop-Free

- It is assumed that the database has no circular definitions.
- Thus, any non-basic unit can be **resolved**, using recursion, to a combination of basic units.
- The number of steps required can be arbitrary.

Example

- Find a normalized quantity representing 1 day:
- > (assoc 'day unicalc-db)
  (day (24 (hour) ()))
- > (assoc 'hour unicalc-db)
  (hour (60 (minute) ()))
- > (assoc 'minute unicalc-db)
  (minute (60 (second) ()))
- > (assoc 'second unicalc-db)
  #f

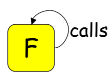
The quantity is ‘(86400 (second)()).

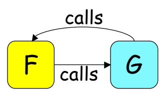
Keep symbolic units sorted

- Easier to divide and multiply (using merge pattern)

Mutual Recursion

- This is recursion wherein one function calls another other than itself, but that function calls the original function, etc.
- Instead of: We have:





Mutual Recursion in Unicalc

- To normalize a Quantity:
  - Normalize each unit in the **numerator**, multiplying those results together.
  - Normalize each unit in the **denominator**, multiplying those results together.
  - **Cancel** units common to the resulting quantity.

Normalizing a Single Unit

- Look up the unit using assoc.
- If the unit is basic (not found), *create* a Quantity for it.

```
> (normalize-unit 'second)
  (1 (second) ())
```

Normalizing a Single Unit

- (Look up the unit using assoc.)
- If the unit is not basic (*was* found), a **numerator** and **denominator** are obtained.
- Each is a list of units.
- Normalize each unit, and *multiply* the results together, then divide the numerator product by the denominator product.
- Finally, combine with the multiplier in the database.

Example

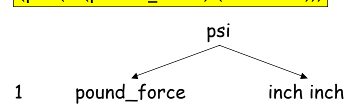
- psi stands for “pounds per square inch

```
> (normalize-unit 'psi)
(6895.006417815267 (kg) (meter second second))
```

- I’m going to conceptualize the normalization process as a tree. This does not imply that you need to *construct* a tree.

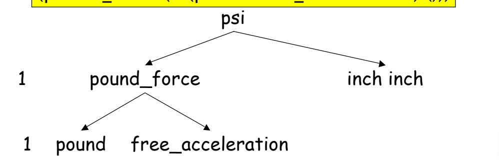
Conceptualization as a Tree

```
> (assoc 'psi unicalc-db)
(psi (1 (pound_force) (inch inch)))
```



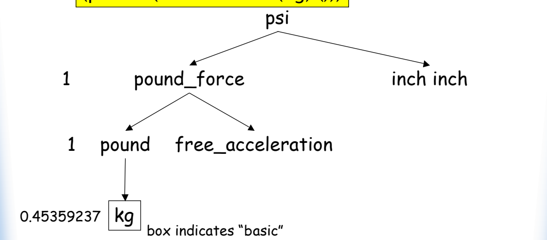
Recursive Expansion

```
> (assoc 'pound_force unicalc-db)
(pound_force (1 (pound free_acceleration) ()))
```



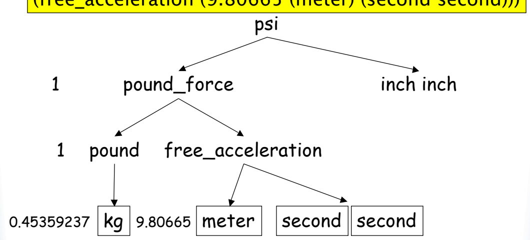
Recursive Expansion

```
> (assoc 'pound unicalc-db)
(pound (0.45359237 (kg) ()))
```



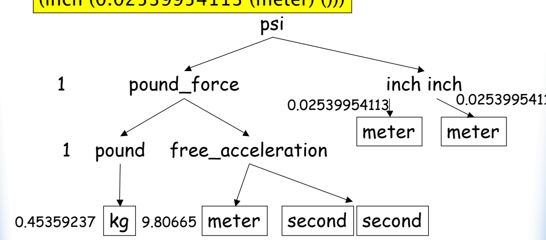
Recursive Expansion

```
> (assoc 'free_acceleration unicalc-db)
(free_acceleration (9.80665 (meter) (second second)))
```

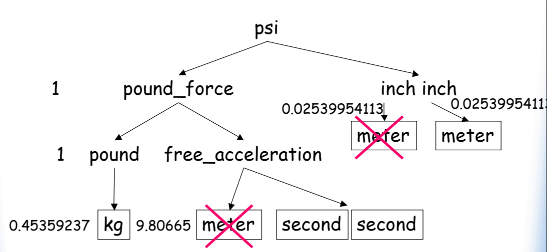


Recursive Expansion

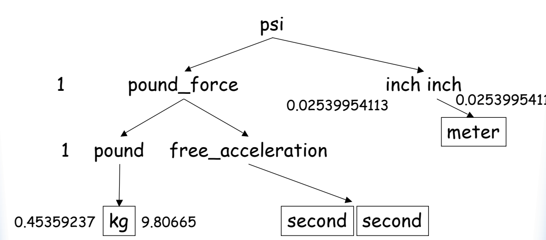
```
> (assoc 'inch unicalc-db)
(inch (0.02539954113 (meter) ()))
```



Cancellation



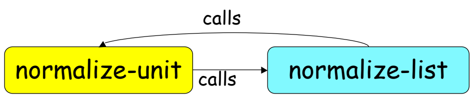
After Cancellation



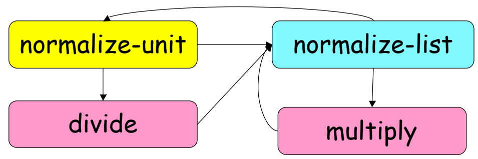
(/ (* 0.45359237 9.80665) (* 0.02539954113 0.02539954113))
= 6895.006417815267 (kg) (meter second second))

Possible Mutual Recursion

- normalize-unit could call a function, say normalize-list (i.e. list of units)



More Mutual Recursion Possibilities (to avoid duplicating work)



Getting Racket to Trace Calls

- Put in your source file:
  (require (lib "trace.rkt"))
- Execute, e.g.:
  (trace normalize-unit …)
- To stop tracing, execute:
  (untrace normalize-unit …)

Example trace Output

```
> (normalize-unit 'psi)
|(normalize-unit psi)
| (normalize-unit pound_force)
| |(normalize-unit pound)
| | (normalize-unit kg)
| | (1 (kg) ())
| |(0.45359237 (kg) ())
| |(normalize-unit free_acceleration)
| | (normalize-unit meter)
| | (1 (meter) ())
| | (normalize-unit second)
| | (1 (second) ())
| | (normalize-unit second)
| | (1 (second) ())
| |(9.80665 (meter) (second second))
| (4.4482216152605 (kg meter) (second second))
```

```
| (normalize-unit inch)
| |(normalize-unit meter)
| |(1 (meter) ())
| (0.02539954113 (meter) ())
| (normalize-unit inch)
| |(normalize-unit meter)
| |(1 (meter) ())
| (0.02539954113 (meter) ())
|(6895.006417815267 (kg) (meter second second))
(6895.006417815267 (kg) (meter second second))
```

result of (normalize-unit pound_force)

Using Auxiliary Functions

- Often the function to be defined is not directly definable in a natural or efficient way using recursion. A **helper** or **auxiliary** function may be necessary.
- Example: reverse

Inefficient Reverse $O(n^2)$

```
(define (reverse L)
  (if (null? L)
      '()
      (append (reverse (rest L))
              (list (first L)))))
```

Use helper function to make efficient reverse O(n)

```
(define (reverse-helper L M)
   (if (null? L)
      M
      (reverse-helper (rest L) (cons (first L) M))))

(define (reverse L) (reverse-helper L '()))
```

M is called an ***accumulator*** argument

Mixed Functional Programming Examples

- Use low-level **or** high-level, whatever fits best
  - Maybe start with low-level, and then use high-level retrospectively, or vice-versa
- Radix conversion
  - Tail recursion
- Tree and graph searching

Tail Recursion

- Sometimes recursive definitions have “cleanup” to be done **after** the recursive call.
- Example:

```
(define (fac N)
   (if (< N 2)
      1
      (* N (fac (- N 1)))))
```

“cleanup” happens after fac call

Stack Build-up

- Functions that require cleanup are non-tail-recursive.
- They build up data on the call stack.
- The depth of recursion could be limited as a consequence.

(fac 4)	(fac 4)	(fac 4)	(fac 4)
	(fac 3)	(fac 3)	(fac 3)
		(fac 2)	(fac 2)
			(fac 1)

arguments kept around for cleanup

Tail Recursion

- Functions that require cleanup are non-tail-recursive.
- They build up data on the stack.
- The depth of recursion could be limited as a consequence.
- Equivalent tail-recursive function (uses a helper):

```
(define (fac N) (fac-helper N 1))
(define (fac-helper N Acc)
   (if (< N 2)
      Acc
      (fac-helper (- N 1) (* N Acc))))
```

no cleanup

happens before call

No Stack Build-up

(fac-helper 4 1)	(fac-helper 3 4)	(fac-helper 2 12)	(fac-helper 1 24)

answer: 24

With tail-recursion, recursive call can be replaced with go-to.

Only certain compilers do this:
Scheme/Racket
Prolog

Others don't or can't:
C++
Java

Convert Number to Binary

- Example:
  - (toBinary 37) ➪
    (1 0 0 1 0 1)
    1*32 + 0*16 + 0*8 + 1*4 + 0*2 + 1*1
    2^5 2^4 2^3 2^2 2^1 2^0
- First try:
  - divide by 2, record remainder, continue with quotient
  - until 0

Convert Number from Binary

- Construct fromBinary, e.g.
  - (fromBinary ‘(1 0 0 1 0 1)) ➪ 37
- Considerations:
  - Do we need an accumulator?
  - Can it be done with tail-recursion?
  - Try it and see.

An Approach

- Write **iterative pseudo-code,** then construct tail-recursive equivalent.
- L = … list to be converted …;
```
Result = 0;
while( L != [ ] )
        {
        Result = 2*Result + first(L);
        L = rest(L);
        }
… answer is in Result …
```

McCarthy’s Transformation

- Any iterative, imperative (i.e. assignment-based) program can be transformed to a system of mutual tail recursions.

Exercises

- Compare “obvious” and tail-recursive forms of:
  - length function
  - sum of a list
  - reduce
  - reverse

Essential Non-Tail Recursions

- Some functions don’t admit a tail-recursive version (unless *reverse* is used before or after):
  - Examples:
    - map, keep, drop
    - append
  - To avoid stack build-up, we might consider using reverse.

append Elimination
(aka “appendectomy”)

- When maximum efficiency is desired, uses of append should be avoided.
- It is often possible to get rid of append by defining versions of functions with an extra accumulator argument.
- Example:

```
(define (nodes Graph)
        (remove_duplicates (append (map first Graph)
                                   (map second Graph)))
```

- Show how to avoid *append* by generalizing map to take an accumulator.

reverse elimination

- Some functions naturally build lists in reverse.
- Rather than immediately reversing the result, consider leaving it as is (in reversed form) and exploiting this fact at a later stage of the functional composition.
- Some functions, such as map, keep, drop, … work equally well whether or not the data is in reverse order.