

Time to learn about NP-completeness!

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Languages

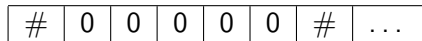
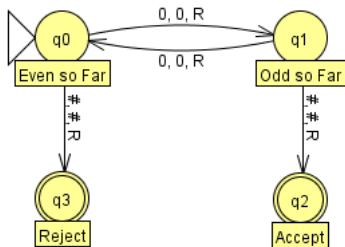
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- Examples
 - The language of strings of all zeros with odd length
 - The language of strings with the same number of zeros and ones

Languages

- A language is a set of strings
- Examples
 - The language of strings of all zeros with odd length
 - The language of strings with the same number of zeros and ones
- If we can figure out whether to accept or reject any particular string as part of a language, we say that that language is *decidable*.

Turing Machines

- One convenient computation model: The Turing Machine
 - Tape containing symbols from an alphabet
 - States
 - Transition rules
- Example: a Turing Machine that decides the language of strings of zeros with odd length.



Non-Deterministic Turing Machines

- Instead of having just one transition rule per state per symbol read on the tape, it may have many
- Allows branching (like running many machines in parallel)
- If any branch reaches the accepting state, then the machine accepts the string

Time Complexity

- Big-Oh notation
 - Counting your toes takes $\mathcal{O}(\text{number of toes})$ time.
 - Adding two n bit numbers takes $\mathcal{O}(n)$ time.
 - Multiplying two $n \times n$ matrices takes $\mathcal{O}(n^3)$ time.
- Language is polynomial time decidable if any string is either accepted or rejected in time proportional to some polynomial in the size of the string

Polynomial Time Reducibility

- Language A is reducible to language B if there is some mapping from strings to strings such the first string is in language A iff the mapping of that string is in language B.
- Silly Example: the language of strings of all zeros of odd length can be reduced to the language of strings of all zeros of even length
- Complicated Example: From the last talk, 3-SAT can be reduced to Graph 3-Colorability

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- Complicated Example: From the last talk, 3-SAT can be reduced to Graph 3-Colorability
- If you can perform this reduction in polynomial time, A is polynomial reducible to B.

\mathcal{P} vs \mathcal{NP}

- \mathcal{P} is the class of languages which can be decided in polynomial time by a deterministic Turing Machine
- \mathcal{NP} is the class of languages which can be decided in polynomial time by a non-deterministic Turing Machine
- Equivalently, \mathcal{NP} is the class of languages which can be verified in polynomial time
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- $\mathcal{P} \subseteq \mathcal{NP}$
- $\mathcal{P} = \mathcal{NP}$?

NP-Completeness

- Two requirements for a language to be \mathcal{NP} -complete
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- Implications of \mathcal{NP} -completeness
 - Polynomial time algorithm for any \mathcal{NP} -complete language yields a polynomial time algorithm for all languages in \mathcal{NP} and means that $\mathcal{P} = \mathcal{NP}$.
 - Worth \$1,000,000

Cook's Theorem - SAT is \mathcal{NP} -complete

- Variables may either be true or false.
- Variables may be NOTed (with \neg), ORed (with \vee), or ANDed (with \wedge).
- Example formula: $(a \vee \neg b \vee c \vee \neg d) \wedge (\neg a \vee c \vee d)$
- SAT is the language which is a collection of formulas such that there is some assignment of variables that makes the formula true.

SAT is \mathcal{NP} -complete

- $\text{SAT} \in \mathcal{NP}$: easy to verify given a correct assignment of variables
- Need to show that any language B in NP can be reduced in polynomial time to SAT
 - Language B can be decided by a non-deterministic Turing Machine in n^k time for some constant k .
 - We can build a huge formula to simulate a Turing Machine running on a string to decide whether it's in language B.

Tableau

#	q_{start}	w_1	w_2	...	w_l	\sqcup	...	\sqcup	#	starting config.						
#									#	second config.						
			<table border="1" style="margin: auto;"> <tr> <td></td> <td></td> <td></td> </tr> <tr> <td></td> <td></td> <td></td> </tr> </table>													
#									#	n^k th config.						

- Essentially a record of the tape and state of the Turing Machine
- Doesn't need to be more than n^k wide
- $\phi = \phi_{\text{cell}} \wedge \phi_{\text{start}} \wedge \phi_{\text{accept}} \wedge \phi_{\text{transition}}$

Proof Sketch

- SAT is in \mathcal{NP}
- Any language in \mathcal{NP} is polynomial time reducible to SAT
 - Languages in \mathcal{NP} are decidable by a non-deterministic Turing Machine in polynomial time
 - Can reduce this machine's running into a formula
 - Formula $\phi = \phi_{\text{start}} \wedge \phi_{\text{cell}} \wedge \phi_{\text{transition}} \wedge \phi_{\text{accept}}$ is satisfiable iff Turing Machine can run to acceptance
 - Reduction in polynomial time

Proof Sketch

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 - Formula $\phi = \phi_{\text{start}} \wedge \phi_{\text{cell}} \wedge \phi_{\text{transition}} \wedge \phi_{\text{accept}}$ is satisfiable iff Turing Machine can run to acceptance
 - Reduction in polynomial time
- So, SAT is \mathcal{NP} -complete!

3-SAT \mathcal{NP} -Complete

- Now that we know SAT is \mathcal{NP} -complete, to show that another is \mathcal{NP} -complete, we only have to show that
 - it's in \mathcal{NP} and
 - that we can reduce SAT to it in polynomial time
- 3-SAT is \mathcal{NP} -Complete
 - 3-SAT is verifiable in polynomial time
 - For every clause $(a_1 \vee a_2 \vee \dots \vee a_l)$ in SAT, make the clauses $(a_1 \vee a_2 \vee z_1) \wedge (\neg z_1 \vee a_3 \vee z_2) \wedge \dots \wedge (\neg z_{l-3} \vee a_{l-1} \vee a_l)$ in 3-SAT

Presentation based on the proof from Prof. Pippenger's Complexity Theory class and Cook's Theorem in Michael Sipser's "Introduction to the Theory of Computation" 2nd ed. 2006.