CS 147:
Computer Systems Performance Analysis
Fractional Factorial Designs

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Overview

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2^{k-p} Designs
Example
Preparing the Sign Table
Confounding
Algebra of Confounding
Design Resolution

Introductory Example of a 2^{k-p} Design

Exploring 7 factors in only 8 experiments:

Run	A	B	C	D	E	F	G
1	-1	-1	-1	1	1	1	-1
2	1	-1	-1	-1	-1	1	1
3	-1	1	-1	-1	1	-1	1
4	1	1	-1	1	-1	-1	-1
5	-1	-1	1	1	-1	-1	1
6	1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1

Analysis of 2^{7-4} Design

- Column sums are zero: $\sum_i x_{ij} = 0 \quad \forall j$
- Sum of 2-column product is zero:

$$\sum_i x_{ij} x_{il} = 0 \quad \forall j \neq l$$

- Sum of column squares is $2^{7-4} = 8$
- Orthogonality allows easy calculation of effects:

$$q_A = \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}$$

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Effects and Confidence Intervals for 2^{k-p} Designs

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$$q_\alpha = \frac{1}{2^{k-p}} \sum_i y_i x_{\alpha i}$$

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  - 2^{k-p} rows and 2^{k-p} columns
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- $2^{k-p} - (k-p) - 1$ interaction (product) columns will remain
- Choose any p columns
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A 2^{4-1} design:

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Example of Preparing a Sign Table

A 2^{4-1} design:

Run	A	B	C	AB	AC	BC	D
1	-1	-1	-1	1	1	1	-1
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3	-1	1	-1	-1	1	-1	1
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Example of Preparing a Sign Table

Why did we choose the ABC column to rename as D? In one sense, the choice is completely arbitrary. But in reality, this leads to a discussion of confounding.

Confounding

- The confounding problem
- An example of confounding
- Confounding notation
- Choices in fractional factorial design

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An Example of Confounding

- Consider this 2^{3-1} table:

I	A	B	C
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- Extend it with an AB column:

I	A	B	C	AB
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There is an animation on this slide

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Analyzing the Confounding Example

- Effect of C is same as that of AB:

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\begin{aligned}
q_C &= (y_1 - y_2 - y_3 + y_4)/4 \\
q_{AB} &= (y_1 - y_2 - y_3 + y_4)/4
\end{aligned}
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- Formula for q_C really gives combined effect:

$$q_C + q_{AB} = (y_1 - y_2 - y_3 + y_4)/4$$

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Confounding Notation

- Previous confounding is denoted by equating confounded effects: $C = AB$
- Other effects are also confounded in this design: $A = BC$, $B = AC$, $C = AB$, $I = ABC$
  - Last entry indicates ABC is confounded with overall mean, or q_0

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Choices in Fractional Factorial Design

- Many fractional factorial designs possible
  - Chosen when assigning remaining p signs
  - 2^p different designs exist for 2^{k-p} experiments
- Some designs better than others
  - Desirable to confound significant effects with insignificant ones
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Rules of Confounding Algebra

- Particular design can be characterized by single confounding
  - Traditionally, use $I = wxyz\ldots$ confounding
- Others can be found by multiplying by various terms
  - I acts as unity (e.g., $I \times A = A$)
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Example: 2^{3-1} Confoundings

- Design is characterized by $I = ABC$
- Multiplying by A gives $A = A^2BC = BC$
- Multiplying by B, C, AB, AC, BC, and ABC:

$$
\begin{aligned}
B &= AB^2C &&= AC \\
C &= ABC^2 &&= AB \\
AB &= A^2B^2C &&= C \\
AC &= A^2BC^2 &&= B \\
BC &= AB^2C^2 &&= A \\
ABC &= A^2B^2C^2 &&= I
\end{aligned}
$$

- Note that only first two lines are unique in this case

Example: 2^{3-1} Confoundings

Generator Polynomials

- Polynomial $I = wxyz\ldots$ is called **generator polynomial** for the confounding
- A 2^{k-p} design confounds 2^p effects together
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Example of Finding Generator Polynomial

- Consider 2^{7-4} design
- Sign table has $2^3 = 8$ rows and columns
- First 3 columns represent A, B, and C
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  - So confoundings are necessarily: $D = AB$, $E = AC$, $F = BC$, and $G = ABC$

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Finishing Generator Polynomial

- Any subset of above terms also multiplies out to I
  - E.g., $ABD \times ACE = A^2BCDE = BCDE$
- Expanding all possible combinations gives 16-term generator (book may be wrong): $I = ABD = ACE = BCF = ABCG = BCDE = ACDF = CDG = ABEF = BEG = AFG = DEF = ADEG = BDFG = CEFG = ABCDEFG$

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- Definition of resolution
- Finding resolution
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- Order of effect is number of factors in it
  - E.g., *I* is order 0, *ABCD* is order 4
- Order of a confounding is sum of effect orders
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