CS 147:
Computer Systems Performance Analysis
Introduction to Queueing Theory

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Overview

Introduction and Terminology
- Poisson Distributions

Fundamental Results
- Stability
- Little's Law

M/M/*
- M/M/1
- M/M/m
- M/M/m/B

More General Queues

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More General Queues

What is a Queueing System?

- A *queueing system* is any system in which things arrive, hang around for a while, and leave
- Examples
  - A bank
  - A freeway
  - A (computer) network
  - A beehive
- The things that arrive and leave are *customers* or *jobs*
- Customers leave after receiving *service*
- Most queueing systems have (surprise!) a *queue* that can store (delay) customers awaiting service

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Parameters of a Queueing System

Arrival Process Injects customers into system
- Usually statistical
- Convenient to specify in terms of *interarrival time* distribution
- Most common is *Poisson* arrivals

Service Time Also statistical

Number of Servers Often 1

System Capacity Equals number of servers plus queue capacity. Often assumed infinite for convenience

Population Maximum number of customers. Often assumed infinite

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Arrival and Service Distributions

- Customer arrivals are random variables
  - Next disk request from many processes
  - Next packet hitting Google
  - Next call to Chipotle
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The Poisson Distribution

- Probability of exactly k arrivals in $(0, t)$ is $\mathrm{P}_k(t) = (\lambda t)^k e^{\lambda t}/k!$
  - λ is *arrival rate* parameter
- More useful formulation is *Poisson arrival distribution*:
  - PDF $A(t) = \mathrm{P}[\text{next arrival takes time} \leq t] = 1 - e^{-\lambda t}$
  - pdf $a(t) = \lambda e^{-\lambda t}$
  - Also known as *exponential* or *memoryless* distribution
  - Mean $=$ standard deviation $= \lambda$
- Poisson distribution is *memoryless*
  - Assume P[arrival within 1 second] at time $t_0 = x$
  - Then P[arrival within 1 second] at time $t_1 > t_0$ is also x
    - I.e., no memory that time has passed
  - Often true in real world
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Splitting and Merging Poisson Processes

- Merging streams of Poisson events (e.g., arrivals) is Poisson

$$\lambda = \sum_{i=1}^{k} \lambda_i$$

- Splitting a Poisson stream randomly gives Poisson streams; if stream i has probability p_i, then

$$\lambda_i = p_i \lambda$$

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Kendall's Notation

A/S/m/B/K/D defines a (single) queueing system compactly:

- *A* Denotes arrival distribution, as follows:
  - *M* Exponential (*M*emoryless)
  - E_k Erlang with parameter k
  - *D* Deterministic
  - *G* Completely general (very hard to analyze!)
- *S* Service distribution, same as arrival
- *m* Number of servers
- *B* System capacity; ∞ if omitted
- *K* Population size; ∞ if omitted
- *D* Service discipline, FCFS if omitted

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Examples of Kendall's Notation

- **D/D/1** Arrivals on clock tick, fixed service times, one server
- **M/M/m** Memoryless arrivals, memoryless service, multiple servers (good model of a bank)
- **M/M/m/m** Customers go away rather than wait in line
- **G/G/1** Modern disk drive

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Common Variables

- τ Interarrival time. Usually varies per customer, e.g., $\tau_1, \tau_2, \ldots$
- λ Mean arrival rate: $1/\overline{\tau}$
- s_i Service time for job i, sometimes called x_i
- μ Mean service rate per server, $1/\overline{s}$
- ρ Traffic intensity or system load = $\lambda/m\mu$. This is the most important parameter in most queueing systems
- w_i Waiting time, or time in queue: interval between arrival and beginning of service
- r_i Response time $= w_i + s_i$

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Stability

- A system is stable iff $\lambda < m\mu \equiv \rho < 1$
- Otherwise, system can't keep up and queue grows to ∞
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Little's Law

- Let $n =$ Number of jobs in system
- Then $n = \lambda \bar{r}$
- Likewise, if $n_q =$ Number of jobs in queue, then $n_q = \lambda \bar{w}$
- True regardless of distributions, queueing disciplines, etc., as long as system is in equilibrium
- May seem obvious:
  - If ten people are ahead of you in line, and each takes about 1 minute for service, you're going to be stuck there for 10 minutes
- Not proved until 1961
- Often useful for calculating queue lengths:
  - Packet takes 2s to arrive, you're sending 100 pps

  $\Rightarrow$ Mean queue length $= 100\ \text{pkt}/\text{s} \times 2\text{s} = 200$ pkts

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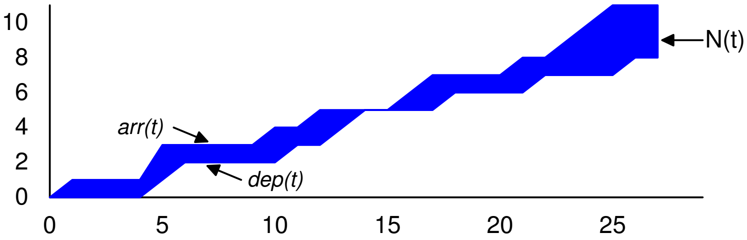
Deriving Little's Law

- Define $arr(t)$ = # of arrivals in interval$(0, t)$
- Define $dep(t)$ = # of departures in interval$(0, t)$
- Clearly, $N(t)$ = # in system at time$t$ = $arr(t) - dep(t)$
- Area between curves = $spent(t)$ = total time spent in system by all customers (measured in customer-seconds)



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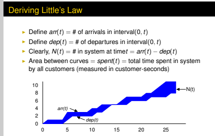
CS147
└Fundamental Results
 └Little's Law
  └Deriving Little's Law



Deriving Little's Law (continued)

- Define average arrival rate during interval t, in customers/second, as $\lambda_t = arr(t)/t$
- Define T_t as system time/customer, averaged over all customers in $(0, t)$
  - Since $spent(t)$ = accumulated customer-seconds, divide by arrivals up to that point to get $T_t = spent(t)/arr(t)$
- Mean tasks in system over $(0, t)$ is accumulated customer-seconds divided by seconds: $\textit{Mean-tasks}_t = spent(t)/t$
- Above three equations give us:

$$\begin{aligned}\textit{Mean-tasks}_t &= spent(t)/t \\ &= T_t arr(t)/t \\ &= \lambda_t T_t\end{aligned}$$

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- We've shown that $\textit{Mean-tasks}_t = \lambda_t T_t$
- Assuming limits of λ_t and T_t exist, limit of $\textit{mean-tasks}_t$ also exists and gives Little's result:

$$\textit{Mean tasks in system} = \textit{arrival rate} \times \textit{mean time in system}$$

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The M/M/1 Queue

- Remember this one if you don't remember anything else
- Assumptions are sometimes realistic, sometimes not
  - Never infinite customers or capacity
  - Service times aren't truly Poisson
  - Interarrival times more likely to be Poisson
- Still provides surprisingly good analysis
- M/M/1's characteristics are clue to many other queues
- Primary results (in equilibrium):
  - Mean number in system $\overline{n} = \rho/(1-\rho)$
  - Mean time in system $\overline{r} = (1/\mu)/(1-\rho) = 1/\mu(1-\rho) = \overline{s}/(1-\rho)$

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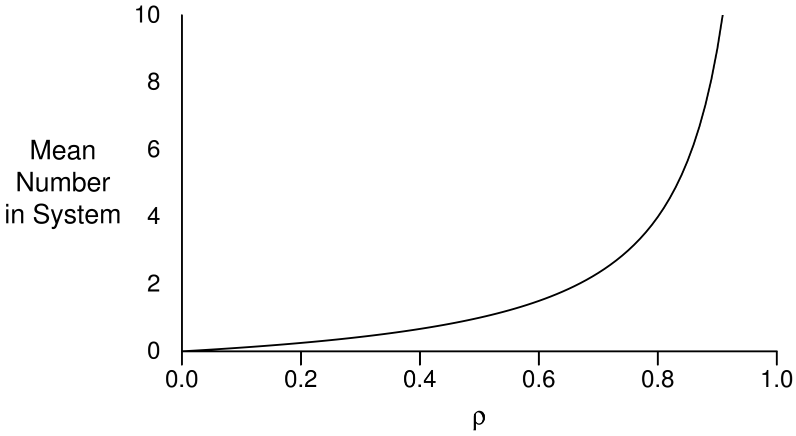
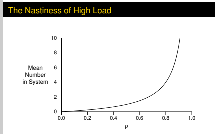
Nearly all useful results in queueing theory apply only to systems in equilibrium.

The Nastiness of High Load





The system breaks down completely at $\rho > 0.95$.

The reason for the breakdown is variance: at high load, a burst fills the queue and it takes a long time to drain, giving plenty of time for another burst to arrive.

More M/M/1 Results

- Variance is $\rho/(1-\rho)^2$, so standard deviation is $\sqrt{\rho}/(1-\rho)$
- q-percentile of time in system is $\bar{r}\ln[100/(100-q)]$
  - 90^{th} percentile is $2.3\bar{r}$
- Mean *waiting* time is $\overline{w} = \frac{1}{\mu}\frac{\rho}{1-\rho}$
- q-percentile of waiting time is

$$\max\left(0, \frac{\overline{w}}{\rho}\ln[100\rho/(100-q)]\right)$$

- Mean jobs served in a busy period: $1/(1-\rho)$
- Probability of n jobs in system $p_n = (1-\rho)\rho^n$
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M/M/1 Example

- Web server gets 1200 requests/hour w/ Poisson arrivals
- Typical request takes 1s to serve
- $\rho = 1200/3600 = 0.33$
- Mean requests in service = $0.33/0.67 = 0.5$
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M/M/1 Example (cont'd)

- Suppose Slashdot raises request rate to 3500/hr
- Now $\rho = 3500/3600 = 0.972$
- Mean requests in service = $0.972/(1 - 0.972) = 34.7$
- $\bar{r} = 1/0.028 = 35.7$ seconds
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M/M/m

- Multiple servers, one queue
- $\rho = \lambda/(m\mu)$
- We'll need probability of empty system:

$$p_0 = \frac{1}{\dfrac{(m\rho)^m}{m!(1-\rho)} + \displaystyle\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!}}$$

- Probability of queueing:

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For $m = 1$, $\varrho = \rho$

M/M/m (cont'd)

- Mean jobs in system: $\overline{n} = m\rho + \rho\varrho/(1-\rho)$
- Mean time in system:

$$\overline{r} = \frac{1}{\mu}\left(1 + \frac{\varrho}{m(1-\rho)}\right)$$

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$m\times$M/M/1 vs. M/M/m

- For m separate M/M/1 queues, each queue sees arrival rate of $\lambda_{M/M/1} = \lambda/m$
  - But ρ is unchanged
- $\overline{r}_{m\times M/M/1} = \frac{1}{\mu}\left(\frac{1}{1-\rho}\right) \overset{?}{>} \overline{r}_{M/M/m} = \frac{1}{\mu}\left(1 + \frac{\varrho}{m(1-\rho)}\right)$
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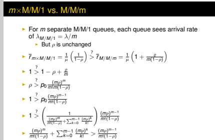

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$m\times$M/M/1 vs. M/M/m

- For m separate M/M/1 queues, each queue sees arrival rate of $\lambda_{M/M/1} = \lambda/m$
  - But ρ is unchanged
- $\overline{r}_{m\times M/M/1} = \frac{1}{\mu}\left(\frac{1}{1-\rho}\right) \overset{?}{>} \overline{r}_{M/M/m} = \frac{1}{\mu}\left(1 + \frac{\varrho}{m(1-\rho)}\right)$
- $1 \overset{?}{>} 1 - \rho + \frac{\varrho}{m}$
- $\rho \overset{?}{>} p_0 \frac{(m\rho)^m}{m!m(1-\rho)}$
- $1 \overset{?}{>} p_0 \frac{(m\rho)^{m-1}}{m!(1-\rho)}$
- $1 \overset{?}{>} \left(\frac{1}{\frac{(m\rho)^m}{m!(1-\rho)} + \sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!}}\right) \frac{(m\rho)^{m-1}}{m!(1-\rho)}$
- $\frac{(m\rho)^m}{m!(1-\rho)} + \sum_{\color{red}{k=0}}^{m-1} \frac{(m\rho)^k}{k!} > \frac{(m\rho)^{m-1}}{m!(1-\rho)}$

2015-06-15

CS147
└M/M/*
  └M/M/m
    └$m\times$M/M/1 vs. M/M/m

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Running Some Numbers

- Assume 5 servers, $\rho = 0.5$, $\mu = 1$
- Then $\bar{r}_{m\times M/M/1} = 1/(1-\rho) = 2$
- $\varrho = \frac{(m\rho)^m}{m!(1-\rho)} p_0 = \frac{(2.5)^5}{5!(0.5)} p_0 = \frac{97.7}{60} p_0 = 1.63 p_0$
- $p_0 = \frac{1}{\frac{(m\rho)^m}{m!(1-\rho)} + \sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!}} = \frac{1}{1.63+1+\frac{2.5^1}{1}+\frac{2.5^2}{2}+\frac{2.5^3}{3!}+\frac{2.5^4}{4!}}$
- $p_0 = \frac{1}{1.63+1+2.5+3.13+2.60+1.63} = \frac{1}{12.49} = 0.08$
- So $\varrho = 1.63(0.08) = 0.13$
- And $\bar{r}_{m/M/m} = 1 + \frac{\varrho}{m(1-\rho)} = 1 + \frac{0.13}{5(1-0.5)} = 1 + \frac{0.13}{2.5} = 1.05$
- In terms of previous slide's inequality, $\frac{97.7}{60}+1+2.5+3.13+2.60+1.63 = 12.49 > \frac{2.5^4}{5!(0.5)} = \frac{39.1}{60} = 0.65$

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$m\times$M/M/1 vs. M/M/m (cont'd)

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- Conclusion: single queue, multiple server is always better than one queue per server
- Question 1: When is this false? (hint: multiple cores)
- Question 2: Why do so many movie theaters have multiple lines for popcorn?

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M/M/m/B

- Real systems have finite capacity
- Previous analysis applies only under light loads (relative to capacity)
- Considering limit has several effects:
  - Lost jobs (obviously)
  - Loss rate p_B becomes important parameter
  - Mean response time drops compared to M/M/m/∞ (Why?)

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Extending the Results

- Unsurprisingly, generality equates to (mathematical) complexity
- Many special cases have been analyzed (e.g., Erlang distributions)
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- Important cases:
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