More Quantum Mechanics and Linear Algebera:
Neutrinos and Other Subatmoic Particles

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Motivation

* Quantum mechanics first formulated as "matrix mechanics"
* Students enjoy tackling "real problems" even if they don't know all the details
* Can combine as many (or few) concepts as wanted in a single problem
* Gives a concrete interpretation to evalues

Truth in Advertising

* The class taught was "mathematical physics"
* All students were math or physics majors
* Multivariable calculus and DEs are co-requisites; a few had taken linear algebra
* First half of the course was on linear algebra
* Material in this talk uses only linear algebra

Quantum Mechanics for the Ph.D. Mathematician

1. Physical states are represented by vectors in a complex Hilbert space
2. Physical observables are represented by self-adjoint operators, and measurements correspond to their spectral projectors
3. ∃ 1-parameter unitary group of evolution operators. Its self-adjoint generator is called the Hamiltonian (eigenvectors are "energy levels"; eigenvalues are "energies")

Quantum Mechanics for Linear Algebra Students

1. Physical states are represented by vectors in a complex vector space

2. Physical observables are represented by Hermitian operators

3. Time evolution is given by the Schrödinger equation:

$$\psi(t) = e^{-\left(\frac{iHt}{\hbar}\right)}\psi(0)$$

   H is called the Hamiltonian, and its eigenvalues are the energies of the system

Neutrino Oscillations

* From late 60's to late 90's, many fewer neutrinos observed than predicted
* "Solar Neutrino Problem"
* "Atmospheric Neutrino Problem"
* Finally resolved by several experiments (1998 to 2004)
    * SuperKamiokande
    * SNO

What's a Neutrino?

* Fundamental particle related to electron, μ, τ
* First hypothesized by Fermi in nuclear decay
* 3 "Flavors" (sometimes "generations")
* Originally believed massless

➡Oscillations indicate non-zero mass!

A Model for Oscillations

* Let $b_1=(1,0)$ represent the electron-type neutrino, $b_2=(0,1)$ the muon-type neutrino.
* The system has an effective Hamiltonian:

$$H = \frac{1}{E_0}\begin{pmatrix} \cos^2(\theta)m_1^2 + \sin^2(\theta)m_2^2 & \cos(\theta)\sin(\theta)\left(m_1^2 - m_2^2\right) \\ \cos(\theta)\sin(\theta)\left(m_1^2 - m_2^2\right) & \sin^2(\theta)m_1^2 + \cos^2(\theta)m_2^2 \end{pmatrix}$$

$$\lambda \in \left\{\frac{m_1^2}{E_0}, \frac{m_2^2}{E_0}\right\}, \; v_\lambda \in \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$$

Observations

* The probability that an electron neutrino be detected as a muon neutrino is

$$P = \left| b_2^T \left(e^{-iHt/\hbar} \right) b_1 \right|^2$$

$$= \sin^2 \left(\frac{\left(m_1^2 - m_2^2\right) t}{2E_0\hbar} \right) \sin^2(2\theta)$$

Example Problem

(20 points) Over the past few years, the phenomenon of neutrino oscillations has been experimentally established. This refers to the fact "electron-type neutrinos" can turn into "muon-type neutrinos". One simple but very accurate model is as follows. Let $e_1 = (1,0)$ represent the electron neutrino, and let $e_2 = (0,1)$ represent the muon neutrino. The particles' evolution is then given by the Hamiltonian:

$$\hat{H} = \begin{pmatrix} m_1^2 \cos^2\theta + m_2^2 \sin^2\theta & \Delta m^2 \cos\theta \sin\theta \\ \Delta m^2 \cos\theta \sin\theta & m_2^2 \cos^2\theta + m_1^2 \sin^2\theta \end{pmatrix}.$$

In the above m_1 and m_2 are two masses (the so-called "electroweak eigenstate masses"), $\Delta m^2 = m_1^2 - m_2^2$, and θ is the *mixing angle* which measures the tendency on one type of neutrino to turn into the other type. If at time $t = 0$ an electron neutrino is created, find the probability that it will be a muon neutrino at a later time t. According to the rules of quantum mechanics, this is given by $P(t) := \left|\left\langle e_2 \,\middle|\, e^{-iHt} e_1 \right\rangle\right|^2$.

Similar Systems

* Particle physics offers many similar systems of this type
  * Cabibbo Angle/CKM Matrix
  * Neutral Kaon system
  * Isospin model of the nucleus
* Why? Particles are idealizations, not fundamental

Group Theory: A Proposal

* Symmetry plays a crucial role modern particle physics (also atomic/molecular)
* Orthogonal and unitary groups most important
* An advanced linear course could give students a systematic introduction.

Examples

* Isospin (SU(2) in fundamental rep)
* The Eight-Fold Way (SU(3) in adjoint and other reps)
* QCD (strong force, SU(3) in adjoint rep)
* Also classical examples: SO(3) in particle mechanics, SO(3,1) in relativity, more.

Why Would You Do Such A Thing?

* Attract more students to advanced linear
* Transition to/reinforce abstract algebra
* Give math students an introduction to continuous groups w/o all the hard topology/geometry
* Service to physics students

Student Response

* “[Schrödinger equation problems] are cool stuff”
* Students found the problems difficult at first but came to enjoy them

Further Reading

* Wikipedia
* Griffiths, David. Introduction to Elementary Particles, Wiley, New York (1987)
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