

Harvey Mudd College
Computer Science 80
Logic for Computer Science
Fall Semester 1999

Assignment #1 – Mathematical Preliminaries
Due 11:00am, Wednesday September 15, 1999

1. Using the directed graph representation used in class, provide examples of relations on a set of four objects that are:
 - (a) transitive, not reflexive, not symmetric
 - (b) transitive, symmetric, not reflexive
 - (c) symmetric, reflexive, not transitive
 - (d) anti-transitive, anti-symmetric, anti-reflexive

2. Define a *non-trivial chain* in a binary relation R as a sequence (a_1, \dots, a_n) for some $n \geq 2$ such that the a_i are distinct, and $(a_i, a_{i+1}) \in R$ for $i \in [n - 1]$. A chain is a *non-trivial cycle* if (a_n, a_1) is also an element of R .

Prove that a relation is a partial order iff it is reflexive and transitive and has no non-trivial cycles.

3. Given two binary relations, R between A and B , and S between B and C , their *composition*, denoted by $R \circ S$, is a relation between A and C defined by the set of ordered pairs $\{(a, c) \mid \text{there is a } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$.
 - (a) Let $R = \{(a, b), (a, c), (c, d), (a, a), (b, a)\}$. What is the value of $R \circ R$?
 - (b) Prove that, given any relation R on a set A , R is transitive iff $R \circ R$ is a subset of R .

4. Given a poset (A, \leq) , define the *lexicographic ordering*, \ll , on $A \times A$, induced by \leq , as follows:

For all $x, y, x', y' \in A$, $(x, y) \ll (x', y')$ iff either:

- $x = x'$ and $y = y'$, or
- $x < x'$, or
- $x = x'$ and $y < y'$

Prove that if the poset (A, \leq) is well-founded, then $(A \times A, \ll)$ is also well-founded. (You may assume [though you might want to confirm for yourself] that if \leq is a partial order, then \ll is indeed a partial order).

5. Ackerman's function on $\mathcal{N} \times \mathcal{N}$ is defined recursively in ML as:

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fun A(x,y) = if x = 0
              then y+1
              else if y = 0
                     then A(x-1,1)
                     else A(x-1, A(x,y-1));
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Assuming that such a recursive definition actually defines a partial function (this is non-trivial but is established in *recursive function theory*), prove by induction over the lexicographic ordering of $\mathcal{N} \times \mathcal{N}$ that Ackerman's function is in fact a total function on pairs of natural numbers. That is, that for any pair of inputs there is a defined output.