

Harvey Mudd College
Computer Science 80
Logic for Computer Science
Fall Semester 1999

Assignment #3 – Propositional Logic, Syntax and Semantics
Sample Solution

1. The proofs in this problem have as an immediate corollary that no proper prefix of a well-formed propositional formula is itself a well-formed propositional formula:

Given a set of propositional letters, P , let Σ be the alphabet of the propositional language over P as defined in class. Define the function $K : \Sigma \rightarrow \mathcal{Z}$ as follows:

- $K(()) = -2, K()) = 2$
- $K(p) = 1$, for all $p \in P \cup \{\perp, \top\}$.
- $K(\neg) = 0$
- $K(\wedge) = K(\vee) = K(\Rightarrow) = K(\Leftarrow) = K(\equiv) = -1$

Further define the function $K' : \Sigma^* \rightarrow \mathcal{Z}$ as $K'(u_1 \dots u_n) = K(u_1) + \dots + K(u_n)$.

- (a) Prove that for any $\Phi \in PROP$, $K'(\Phi) = 1$.

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We proceed by complete induction on the length of the formula.

Suppose $|\Phi| = n$ and, for all well-formed formulas Ψ such that $|\Psi| < n$, we know that $K'(\Psi) = 1$.

There are two cases:

$|\Phi| = 1$: In that case, since Φ is a wff, then $\Phi = p$ for some $p \in P \cup \{\perp, \top\}$. But then, by definition, $K'(\Phi) = K'(p) = K(p) = 1$.

$|\Phi| > 1$: In that case, since Φ is a wff, then either $\Phi = (\neg\Psi)$ for some wff Ψ , or $\Phi = (\Psi_1 \diamond \Psi_2)$ for some wffs Ψ_1, Ψ_2 and some $\diamond \in \{\wedge, \vee, \Rightarrow, \Leftarrow, \equiv\}$. We will deal we these two subcases individually.

$\Phi = (\neg\Psi)$ **for some wff Ψ** : In that case, clearly $|\Psi| < n$, and, by the induction hypothesis, $K'(\Psi) = 1$. Suppose that $\Psi = u_1 \dots u_m$. Then $\Phi = (\neg u_1 \dots u_m)$, and

$$\begin{aligned} K'(\Phi) &= K(() + K(\neg) + K(u_1) + \dots + K(u_m) + K()) \\ &= K(() + K(\neg) + K'(\Psi) + K()) \\ &= -2 + 0 + 1 + 2 \\ &= 1 \end{aligned}$$

$\Phi = (\Psi_1 \diamond \Psi_2)$ for some wffs Ψ_1, Ψ_2 and some $\diamond \in \{\wedge, \vee, \Rightarrow, \Leftarrow, \equiv\}$: In that case, clearly $|\Psi_1| < n$ and $|\Psi_2| < n$, and, by the induction hypothesis, $K'(\Psi_1) = K'(\Psi_2) = 1$. Further, suppose that $\Psi_1 = u_{11} \cdots u_{1m}$ and $\Psi_2 = u_{21} \cdots u_{2l}$. Then $\Phi = (u_{11} \cdots u_{1m} \diamond u_{21} \cdots u_{2l})$, and

$$\begin{aligned} K'(\Phi) &= K(\text{()}) + K(u_{11}) + \cdots + K(u_{1m}) + K(\diamond) + \\ &\quad K(u_{21}) + \cdots + K(u_{2l}) + K(\text{()}) \\ &= K(\text{()}) + K'(\Psi_1) + K(\diamond) + K'(\Psi_2) + K(\text{()}) \\ &= -2 + 1 + -1 + 1 + 2 \\ &= 1 \end{aligned}$$

So, in every case, we have shown that $K'(\Phi) = 1$.

Q.E.D.

- (b) (extra credit) Prove that for any $w \in \Sigma^*$ such that there is a $\Phi \in PROP$ such that w is a proper prefix of Φ , $K'(w) \neq 1$.

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We will, in fact, prove a stronger fact, that for any $w \in \Sigma^*$ such that there is a $\Phi \in PROP$ such that w is a proper prefix of Φ , $K'(w) < 1$. This is necessary to make the inductive argument go through.

While it may seem counter-intuitive, we will, again, proceed by induction on the length of Φ , ignoring the length of w .

Consider a particular arbitrary Φ and w and assume that the theorem holds for all proper prefixes of any wff Ψ such that $|\Psi| < |\Phi|$

We now examine by cases based on the structure of Φ . Since w must be a proper prefix of Φ , $|\Phi|$ must be greater than 1, so we need not consider the cases where Φ is a propositional letter or truth constant.

$\Phi = (\neg\Psi)$ for some wff Ψ : Then w can be of one of three forms:

$$w = (\text{:} \text{ Then } K'(w) = K(\text{()}) = -2$$

$$w = (\neg\text{:} \text{ Then } K'(w) = K(\text{()}) + K(\neg) = -2 + 0 = -2$$

$$w = (\neg\Psi\text{:} \text{ Then, assuming } \Psi = u_1 \cdots u_m, \text{ we know by the previous problem that } K'(\Psi) = 1 \text{ and we have that:}$$

$$\begin{aligned} K'(w) &= K(\text{()}) + K(\neg) + K(u_1) + \cdots + K(u_m) \\ &= K(\text{()}) + K(\neg) + K'(\Psi) \\ &= -2 + 0 + 1 \\ &= -1 \end{aligned}$$

$w = (\neg w')$ for some w' a proper prefix of Ψ : Then, we know by the induction hypothesis (since $|\Psi| < |\Phi|$) that $K'(w') < 1$. Suppose that

$w' = u_1 \cdots u_{m'}$ and that $K'(w') = x$. Then, we have that:

$$\begin{aligned}
K'(w) &= K(\text{)} + K(\neg) + K(u_1) + \cdots + K(u'_{m'}) \\
&= K(\text{)} + K(\neg) + K'(w') \\
&= -2 + 0 + x \\
&< -2 \quad (\text{since } x < 1)
\end{aligned}$$

$\Phi = (\Psi_1 \diamond \Psi_2)$ **for some wffs Ψ_1, Ψ_2 and some $\diamond \in \{\wedge, \vee, \Rightarrow, \Leftarrow, \equiv\}$** : Then w can be of one of six forms. (We will give a more abbreviated justification of each case than in the previous step.):

$w = (\text{:}$ Then, as above, $K'(w) = K(\text{)} = -2$.

$w = (\Psi_1$: Then, $K'(w) = K(\text{)} + K'(\Psi) = -2 + 1 = -1$.

$w = (\Psi_1 \diamond$: Then, $K'(w) = K(\text{)} + K'(\Psi_1) + K(\diamond) = -2 + 1 + -1 = -2$.

$w = (\Psi_1 \diamond \Psi_2$: Then, $K'(w) = K(\text{)} + K'(\Psi_1) + K(\diamond) + K'(\Psi_2) = -2 + 1 + -1 + 1 = -1$.

$w = (w'$ **for some w' a proper prefix of Ψ_1** : Then, as above, by the induction hypothesis we may assume that $K(w') = x < 1$, and, therefore, $K'(w) = K(\text{)} + K'(w') = -2 + x < -2$.

$w = (\Psi_1 \diamond w'$ **for some w' a proper prefix of Ψ_2** : Then, as above, by the induction hypothesis we may assume that $K(w') = x < 1$, and, therefore, $K'(w) = K(\text{)} + K'(\Psi_1) + K(\diamond) + K'(w') = -2 + 1 + -1 + x = -2 + x < -2$.

So, in every case, we have shown that $K'(w) < 1$.

Q.E.D.