Harvey Mudd College
Computer Science 80
Logic for Computer Science
Spring Semester 1999

Assignment #4 – Propositional Logic: Hilbert Systems
Sample Solution

Prove each of the following derived Hilbert rules sound by showing that when you have a proof of each of the premises (the formulas on the top) you can construct a proof of the conclusion (the formula on the bottom) without using the rule.

You may uses any of the derived rules covered in class, and any of the rules you have already proved.

You will find the framework of most of these proofs in section 2.9 of the text, but in a different style than used in class.

1. In class, we gave a derivation justifying the following rule which used the deduction rule. Please give a derivation that uses only the pure Hilbert system (consisting of the three axioms and *modus ponens*).

   The Transitivity Rule:

$$\frac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}$$

• • •

   To be filled in later.

2. The Inverse Contrapositive Rule:

$$\frac{A \Rightarrow B}{(\neg B) \Rightarrow (\neg A)}$$

• • •

   The following is the solution given in the text. A shorter solution follows.

   Note: The book refers to $\neg B \Rightarrow (B \Rightarrow \neg\neg B)$ as an instance of Theorem 2.9.8, and, as such, leaves it as an unexpanded leaf. Here, we first expand its proof for use within the larger proof.

$$\dfrac{\dfrac{\not B^{2} \qquad \dfrac{\dfrac{\not{\neg B}^{1} \qquad \dfrac{}{\neg B \Rightarrow (\neg\neg\neg B \Rightarrow \neg B)}\,{\scriptstyle axiom\ 1}}{\neg\neg\neg B \Rightarrow \neg B}\,{\scriptstyle MP} \qquad \dfrac{}{(\neg\neg\neg B \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg\neg B)}\,{\scriptstyle axiom\ 3}}{B \Rightarrow \neg\neg B}\,{\scriptstyle MP}}{\dfrac{\neg\neg B}{B \Rightarrow \neg\neg B}\,{\scriptstyle DR2}}\,{\scriptstyle MP}}{\neg B \Rightarrow (B \Rightarrow \neg\neg B)}\,{\scriptstyle DR1}$$

$$\dfrac{\not{\neg B}^{1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\not{\neg\neg A}^{2}}{A}\,\text{double }\neg \qquad \begin{matrix}\vdots\\ A \Rightarrow B\end{matrix}}{B}\,{\scriptstyle MP} \qquad \dfrac{\not{\neg B}^{1} \qquad \dfrac{}{\neg B \Rightarrow (B \Rightarrow \neg\neg B)}\,{\scriptstyle thm\ 2.9.8}}{B \Rightarrow \neg\neg B}\,{\scriptstyle MP}}{\neg\neg B}\,{\scriptstyle MP}}{\neg\neg A \Rightarrow \neg\neg B}\,{\scriptstyle DR2}}{\neg B \Rightarrow \neg A}\,{\scriptstyle contrapositive}}{\dfrac{\neg A}{\neg B \Rightarrow \neg A}\,{\scriptstyle DR1}}\,{\scriptstyle MP}$$

If we note that the solution to the next problem (Inverse Double Negation) does not make use of Inverse Contrapositive, we can assume that we already have the IDN rule and use it to obtain a much shorter proof as below. Even if we do not assume IDN, we could expand the IDN step and still obtain a shorter proof than the one above (particulalry if we use the second, shorter expansion of IDN).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\not{\neg\neg A}^{1}}{A}\,{\scriptstyle DN} \qquad \begin{matrix}\vdots\\ A \Rightarrow B\end{matrix}}{B}\,{\scriptstyle MP}}{\neg\neg B}\,{\scriptstyle IDN}}{\neg\neg A \Rightarrow \neg\neg B}\,{\scriptstyle DR1}}{\neg B \Rightarrow \neg A}\,{\scriptstyle contrapositive}$$

3. The Inverse Double Negation Rule:

$$\frac{A}{\neg\neg A}$$

• • •

The following is the solution given in the text. A shorter solution follows.

$$\dfrac{\begin{matrix}\vdots\\ A\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\neg\neg\neg\not{A}^1 \qquad \dfrac{\dfrac{\dfrac{\neg\neg\neg\not{A}^1 \qquad \dfrac{}{\neg\neg A \Rightarrow (\neg\neg\neg\neg A \Rightarrow \neg\neg A)}\ \textit{axiom 1}}{\neg\neg\neg\neg\neg\neg A \Rightarrow \neg\neg\neg A}\ {\scriptstyle MP}}{\neg\neg A \Rightarrow \neg\neg\neg\neg\neg A}\ \textit{contrapositive}}{\neg\neg\neg A \Rightarrow \neg A}\ \textit{contrapositive}}{\neg A}\ {\scriptstyle MP}}{\neg\neg\neg A \Rightarrow \neg A}\ {\scriptstyle DR}1}{A \Rightarrow \neg\neg A}\ \textit{contrapositive}}{\neg\neg A}\ {\scriptstyle MP}$$

It is not clear to me why the text author went the roundabout route of proving the $\neg A$ from $\neg\neg\neg A$ using MP (and hence having to prove the other premise of the MP rule) when he could much more simply have used ordinary double negation to conclude it:

$$\dfrac{\begin{matrix}\vdots\\ A\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\neg\neg\neg\not{A}^1}{\neg A}\ {\scriptstyle DN}}{\neg\neg\neg A \Rightarrow \neg A}\ {\scriptstyle DR}1}{A \Rightarrow \neg\neg A}\ \textit{contrapositive}}{\neg\neg A}\ {\scriptstyle MP}$$

4. The And Rule:

$$\dfrac{A \quad B}{A \wedge B}$$

• • •

Since there is no $\wedge$ in the system, we will derive the equivalent rule:

$$\dfrac{A \quad B}{\neg(A \Rightarrow \neg B)}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ B\end{matrix}}{\neg\neg B}\ {\scriptstyle IDN} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix} \qquad \dfrac{\dfrac{A \not\Rightarrow \neg B^1}{(A \Rightarrow \neg B) \Rightarrow (A \Rightarrow \neg B)}\ {\scriptstyle DR}1}{A \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg B)}\ \textit{Exch. of Ant.}}{(A \Rightarrow \neg B) \Rightarrow \neg B}\ {\scriptstyle MP}}{\neg\neg B \Rightarrow \neg(A \Rightarrow \neg B)}\ {\scriptstyle ICP}}{\neg(A \Rightarrow \neg B)}\ {\scriptstyle MP}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ B\end{matrix}}{\neg\neg B}\ {\scriptstyle IDN} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix} \qquad A \not\Rightarrow \neg B^1}{\neg B}\ {\scriptstyle MP}}{(A \Rightarrow \neg B) \Rightarrow \neg B}\ {\scriptstyle MP}}{\neg\neg B \Rightarrow \neg(A \Rightarrow \neg B)}\ {\scriptstyle ICP}}{\neg(A \Rightarrow \neg B)}\ {\scriptstyle MP}$$

5. The Commutativity of Or Rule:

$$\frac{B \vee A}{A \vee B}$$

•••

Since there is no $\vee$ in the system, we will derive the equivalent rule:

$$\frac{\neg B \Rightarrow A}{\neg A \Rightarrow B}$$

$$\dfrac{\dfrac{\dfrac{\neg \not{A}^{1} \qquad \dfrac{\begin{array}{c}\vdots \\ \neg B \Rightarrow A\end{array}}{\neg A \Rightarrow \neg\neg B}\text{ contrapositive}}{\neg\neg B}\ {\scriptstyle MP}}{B}\text{ double } \neg}{\neg A \Rightarrow B}\ {\scriptstyle DR}1$$