Harvey Mudd College
Computer Science 80
Logic for Computer Science
Fall Semester 1999

Assignment #5 – Propositional Logic: Natural Deduction
Sample Solution

1. Give Natural Deduction proofs of each of the following formulas (brackets are used in place of parentheses in some formulas to make the structure clearer):

   (a) $[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]$

   Note that in the answers to this problem, any leaf with a superscripted number attached is a discharged assumption, and should be assumed to be crossed out. (I could not figure out how to get LaTeX to cross out large expressions.)

$$\dfrac{\dfrac{\dfrac{q \wedge p^{2}}{q}\wedge_E \quad \dfrac{\dfrac{q \wedge p^{2}}{p}\wedge_E \quad p \Rightarrow (q \Rightarrow r)^{1}}{q \Rightarrow r}\Rightarrow_E}{\dfrac{r}{(p \wedge q) \Rightarrow r}\Rightarrow_I 2} \Rightarrow_E \quad \dfrac{\dfrac{\not{p}^{2} \quad \not{q}^{3}}{p \wedge q}\wedge_I \quad (p \wedge q) \Rightarrow r^{1}}{\dfrac{r}{q \Rightarrow r}\Rightarrow_I 3}\Rightarrow_E}{\dfrac{p \Rightarrow (q \Rightarrow r)}{[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]}\equiv_I}\Rightarrow_I 2$$

   (b) $(p \Rightarrow q) \Rightarrow [(r \Rightarrow p) \Rightarrow (r \Rightarrow q)]$

$$\dfrac{\dfrac{\dfrac{\not{r}^{3} \quad r \Rightarrow p^{2}}{p}\Rightarrow_E \quad p \Rightarrow q^{1}}{q}\Rightarrow_E}{\dfrac{(r \Rightarrow p) \Rightarrow (r \Rightarrow q)}{(p \Rightarrow q) \Rightarrow [(r \Rightarrow p) \Rightarrow (r \Rightarrow q)]}\Rightarrow_I 2}\Rightarrow_I 1$$

In two of the next three proofs we require a proof of the formula $(p \vee \neg p)$. This proof, for any formula p, has the form:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\not{p}^{2}}{p \vee \neg p}\vee_I \quad \neg(p \vee \neg p)^{1}}{\dfrac{\bot}{\not{p}}\,RAA\ 2}\bot_I \quad \dfrac{\dfrac{\dfrac{\not{p}^{3}}{p \vee \neg p}\vee_I \quad \neg(p \vee \neg p)^{1}}{\dfrac{\bot}{\neg p}\,\neg_I 2}\bot_I}{\bot}\,RAA\ 3}{p \vee \neg p}\,RAA\ 1$$

It is elided as $\vdots$ in the proofs below to save space.

(c) $([(p \Rightarrow q) \Rightarrow p] \Rightarrow p)$

$$\dfrac{\begin{array}{c}\vdots\\ p \vee \neg p\end{array} \quad \dfrac{\dfrac{\dfrac{\dfrac{\neg p^{2} \quad \not{p}^{3}}{\bot}\bot_I}{\not{q}}\bot_E}{p \Rightarrow q}\Rightarrow_I 3 \quad \dfrac{(p \Rightarrow q) \Rightarrow p^{1}}{p}\Rightarrow_E \quad \not{p}^{2}}{p}\vee_E 2}{[(p \Rightarrow q) \Rightarrow p] \Rightarrow p}\Rightarrow_I 1$$

This could also be done without the use of $p \vee \neg p$ as:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg p^{2} \quad \not{p}^{3}}{\bot}\bot_I}{\not{q}}\bot_E}{p \Rightarrow q}\Rightarrow_I 3 \quad (p \Rightarrow q) \Rightarrow p^{1}}{p} \quad \dfrac{\neg p^{2}}{\bot}\neg_E}{\dfrac{\bot}{p}\,RAA\ 2}}{[(p \Rightarrow q) \Rightarrow p] \Rightarrow p}\Rightarrow_I 1$$

(d) $(p \Rightarrow r) \equiv \neg(p \wedge \neg r)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\not{p}^{2} \quad \neg r^{3}}{p \wedge \neg r}\wedge_I}{\bot}\neg_E \quad \neg(p \wedge \neg r)^{1}}{\dfrac{r}{p \Rightarrow r}\Rightarrow_I 2}\,RAA 3 \quad \dfrac{\dfrac{p \wedge \neg r^{2}}{\neg r}\wedge_E \quad \dfrac{\dfrac{p \wedge \neg r^{2}}{p}\wedge_E \quad p \Rightarrow r^{1}}{r}\Rightarrow_E}{\bot}\bot_I}{\dfrac{\bot}{\neg(p \wedge \neg r)}\neg_I 2}}{(p \Rightarrow r) \equiv \neg(p \wedge \neg r)}\equiv_I 1$$

(e) $(p \Rightarrow r) \equiv (\neg p \vee r)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\not{p}^{2} \quad \neg p^{3}}{\bot}\bot_I}{\not{r}}\bot_E}{r}\vee_E 3 \quad \dfrac{\begin{array}{c}\vdots\\ p \vee \neg p\end{array} \quad \dfrac{\dfrac{\not{p}^{2} \quad p \Rightarrow r^{1}}{r}\Rightarrow_E}{\neg p \vee r}\vee_I \quad \dfrac{\neg p^{2}}{\neg p \vee r}\vee_I}{\neg p \vee r}\vee_E 2}{p \Rightarrow r}\Rightarrow_I 2}{(p \Rightarrow r) \equiv (\neg p \vee r)}\equiv_I 1$$