

Harvey Mudd College  
Computer Science 80  
Logic for Computer Science  
Fall Semester 1999

Assignment #6 – Propositional Logic: Gentzen Sequent Calculus  
**Sample Solution**

1. Give Gentzen Sequent Calculus proofs of each of the following formulas:

(a)  $[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]$

$$\frac{\frac{\frac{\overline{q, p \longrightarrow r, p} \text{ id}}{q, p \longrightarrow r, p \wedge q} \wedge_R \quad \frac{\overline{q, p \longrightarrow r, q} \text{ id}}{q, p, r \longrightarrow r} \Rightarrow_L}{\frac{q, p, (p \wedge q) \Rightarrow r \longrightarrow r}{p, (p \wedge q) \Rightarrow r \longrightarrow q \Rightarrow r} \Rightarrow_R} \Rightarrow_R}{(p \wedge q) \Rightarrow r \longrightarrow p \Rightarrow (q \Rightarrow r)} \Rightarrow_R \quad \frac{\frac{\frac{\overline{p, q \longrightarrow r, q} \text{ id}}{p, q, q \Rightarrow r \longrightarrow r} \Rightarrow_L \quad \frac{\overline{p, q, r \longrightarrow r} \text{ id}}{p, q, q \Rightarrow r \longrightarrow r} \Rightarrow_L}{\frac{p, q, p \Rightarrow (q \Rightarrow r) \longrightarrow r}{p \wedge q, p \Rightarrow (q \Rightarrow r) \longrightarrow r} \wedge_L} \Rightarrow_R}{p \Rightarrow (q \Rightarrow r) \longrightarrow (p \wedge q) \Rightarrow r} \equiv_R}{\longrightarrow [(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]}$$

(b)  $(p \Rightarrow q) \Rightarrow [(r \Rightarrow p) \Rightarrow (r \Rightarrow q)]$

$$\frac{\frac{\frac{\overline{r \longrightarrow q, p, r} \text{ id}}{r, r \Rightarrow p \longrightarrow q, p} \Rightarrow_L \quad \frac{\overline{r, p \longrightarrow q, p} \text{ id}}{r, r \Rightarrow p, q \longrightarrow q} \Rightarrow_L}{\frac{r, r \Rightarrow p, p \Rightarrow q \longrightarrow q}{r \Rightarrow p, p \Rightarrow q \longrightarrow r \Rightarrow q} \Rightarrow_R} \Rightarrow_R}{p \Rightarrow q \longrightarrow (r \Rightarrow p) \Rightarrow (r \Rightarrow q)} \Rightarrow_R}{\longrightarrow (p \Rightarrow q) \Rightarrow [(r \Rightarrow p) \Rightarrow (r \Rightarrow q)]}$$

(c)  $[(p \Rightarrow q) \Rightarrow p] \Rightarrow p$

$$\frac{\frac{\frac{\overline{p \longrightarrow q, p} \text{ id}}{\longrightarrow p \Rightarrow q, p} \Rightarrow_R \quad \frac{\overline{p \longrightarrow p} \text{ id}}{p \longrightarrow p} \Rightarrow_L}{(p \Rightarrow q) \Rightarrow p \longrightarrow p} \Rightarrow_R}{\longrightarrow ((p \Rightarrow q) \Rightarrow p) \Rightarrow p}$$

(d)  $(p \Rightarrow r) \equiv \neg(p \wedge \neg r)$

$$\frac{\frac{\frac{\overline{p, \neg r \longrightarrow p} \text{ id}}{p, \neg r, p \Rightarrow r \longrightarrow} \wedge_L \quad \frac{\overline{p, r \longrightarrow r} \text{ id}}{p, \neg r, r \longrightarrow} \neg_L}{\frac{p, \neg r, p \Rightarrow r \longrightarrow}{p \wedge \neg r, p \Rightarrow r \longrightarrow} \wedge_L} \neg_R}{p \Rightarrow r \longrightarrow \neg(p \wedge \neg r)} \neg_R \quad \frac{\frac{\frac{\overline{p \longrightarrow r, p} \text{ id}}{p \longrightarrow r, p \wedge \neg r} \wedge_R \quad \frac{\overline{r, p \longrightarrow r} \text{ id}}{p \longrightarrow r, \neg r} \neg_R}{\frac{p, \neg(p \wedge \neg r) \longrightarrow r}{\neg(p \wedge \neg r) \longrightarrow p \Rightarrow r} \Rightarrow_R} \wedge_R}{\neg(p \wedge \neg r) \longrightarrow p \Rightarrow r} \equiv_R}{\longrightarrow (p \Rightarrow r) \equiv \neg(p \wedge \neg r)}$$



2. Consider the following variation on the sequent calculus in which the right hand side of the sequent is restricted to a single fomula (Greek letters denote sets of formulas, roman letters denote individual formulas):

$$\begin{array}{c} \overline{\Gamma, A \longrightarrow A} \textit{id} \quad \overline{\Gamma, \perp \longrightarrow A} \perp \\ \\ \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \wedge_L \quad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge_R \\ \\ \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C} \vee_L \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B} \vee_{R1} \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B} \vee_{R2} \\ \\ \frac{\Gamma \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \Rightarrow B \longrightarrow C} \Rightarrow_L \quad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \Rightarrow B} \Rightarrow_R \\ \\ \frac{\Gamma \longrightarrow A}{\Gamma, \neg A \longrightarrow C} \neg_L \quad \frac{\Gamma, A \longrightarrow \perp}{\Gamma \longrightarrow \neg A} \neg_R \end{array}$$

Prove that this version of the sequent calculus is sound (for the fragment of propositional calculus it covers) by showing that whenever there is a derivation of a sequent  $\Gamma \longrightarrow A$  in this system, then there is a Natural Deduction proof of the formula  $A$  from open assumptions  $\Gamma$ .

(Hint: Use complete induction on the height of the sequent calculus proof.)

**Proof:** The proof is by complete induction on the height of the intuitionistic Sequent Calculus proof. We assume that for all sequent calculus proofs of height less than  $n$  that the theorem holds. We now show that the theorem holds for any sequent calculus proof of height  $n$ . The proof is by cases based on the final (i.e. bottom) rule of the Sequent Calculus proof.

- Suppose the last (and only) rule applied is of the form:

$$\overline{\Gamma, A \longrightarrow A} \textit{id}$$

Then we need to show that there is a Natural Deduction proof of  $A$  from open assumptions in the set  $\Gamma \cup \{A\}$ . But:

$A$

is such a proof (in which  $A$  is both the conclusion and an open assumption).

- Suppose the last (and only) rule applied is of the form:

$$\frac{}{\Gamma, \perp \longrightarrow A} \textit{id}$$

Then we need to show that there is a Natural Deduction proof of  $A$  from open assumptions in the set  $\Gamma \cup \{\perp\}$ . But:

$$\frac{\perp}{A} \perp_E$$

is such a proof (in which  $A$  is the conclusion and  $\perp$  is the only open assumption).

- Suppose the last rule applied is:

$$\frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \wedge_L$$

Then we need to show that there is a Natural Deduction proof of  $C$  from open assumptions in the set  $\Gamma \cup \{A \wedge B\}$ . But, by the induction hypothesis, since the proof of the upper sequent is shorter than the overall proof, there is a proof of  $C$  from open assumptions in the set  $\Gamma \cup \{A, B\}$  of the form:

$$\begin{array}{c} \Gamma \quad A \quad B \\ \vdots \\ C \end{array}$$

But then we may cap any leaves labeled with the propositions  $A$  and  $B$  with applications of the  $\wedge_E$  rule, as in:

$$\Gamma \quad \frac{A \wedge B}{A} \wedge_E \quad \frac{A \wedge B}{B} \wedge_E \\ \vdots \\ C$$

yielding a proof of the desired form.

(Note, that it is possible that  $A$ , or  $B$ , or both do not actually appear among the leaves of the Natural Deduction proof from the induction hypothesis. In that case, the construction simply omits the application of  $\wedge_E$  for that proposition, and the result still holds. This behavior will be assumed in the rest of the cases.)

- Suppose the last rule applied is:

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge_R$$

Then we need to show that there is a Natural Deduction proof of  $A \wedge B$  from open assumptions in the set  $\Gamma$ . But, by the induction hypothesis, there are proofs of  $A$  and  $B$  from open assumptions in the set  $\Gamma$  of the form:

$$\begin{array}{ccc} \Gamma & & \Gamma \\ \vdots & & \vdots \\ A & \text{and} & B \end{array}$$

But then we may join those two proofs with an application of the  $\wedge_I$  rule, as in:

$$\frac{\begin{array}{c} \Gamma \quad \Gamma \\ \vdots \quad \vdots \\ \underline{A \quad B} \end{array}}{A \wedge B} \wedge_I$$

yielding a proof of the desired form.

- Suppose the last rule applied is:

$$\frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C} \vee_L$$

Then we need to show that there is a Natural Deduction proof of  $C$  from open assumptions in the set  $\Gamma \cup \{A \vee B\}$ . But, by the induction hypothesis, there are proofs of  $C$  from open assumptions in the set  $\Gamma \cup \{A\}$  and from open assumptions in the set  $\Gamma \cup \{B\}$  of the form:

$$\begin{array}{c} \Gamma \quad A \\ \vdots \\ C \end{array} \quad \text{and} \quad \begin{array}{c} \Gamma \quad B \\ \vdots \\ C \end{array}$$

But then we may join those two proofs with an application of the  $\vee_E$  rule, as in:

$$\frac{\begin{array}{c} \Gamma \quad A \quad \Gamma \quad B \\ \vdots \quad \quad \quad \vdots \\ \underline{A \vee B \quad C \quad C} \end{array}}{C} \vee_E$$

yielding a proof of the desired form.

- Suppose the last rule applied is:

$$\frac{\Gamma \longrightarrow A_i}{\Gamma \longrightarrow A_1 \vee A_2} \vee_{R2}$$

Then we need to show that there is a Natural Deduction proof of  $A_1 \vee A_2$  from open assumptions in the set  $\Gamma$ . But, by the induction hypothesis, there is a proof of  $A_i$  (for some  $i \in \{1, 2\}$ ) from open assumptions in the set  $\Gamma$  of the form:

$$\begin{array}{c} \Gamma \\ \vdots \\ A_i \end{array}$$

But then we may terminate the proof with an application of the  $\vee_I$  rule, as in:

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ A_i \end{array}}{A_1 \vee A_2} \vee_I$$

yielding a proof of the desired form.

- Suppose the last rule applied is:

$$\frac{\Gamma \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \Rightarrow B \longrightarrow C} \Rightarrow_L$$

Then we need to show that there is a Natural Deduction proof of  $C$  from open assumptions in the set  $\Gamma \cup \{A \Rightarrow B\}$ . But, by the induction hypothesis, there are proofs of  $A$  from open assumptions in the set  $\Gamma$  and of  $C$  from open assumptions in the set  $\Gamma \cup \{B\}$  of the form:

$$\begin{array}{c} \Gamma \\ \vdots \\ A \end{array} \quad \text{and} \quad \begin{array}{c} \Gamma \quad B \\ \vdots \\ C \end{array}$$

But then we may cap any leaves of the proof of  $C$  that are labeled with the proposition  $B$  with an application of the  $\Rightarrow_E$  rule, as in:

$$\begin{array}{c} \Gamma \\ \vdots \\ \Gamma \quad \frac{A \quad A \Rightarrow B}{B} \Rightarrow_E \\ \vdots \\ C \end{array}$$

yielding a proof of the desired form.

- Suppose the last rule applied is:

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \Rightarrow B} \Rightarrow_R$$

Then we need to show that there is a Natural Deduction proof of  $A \Rightarrow B$  from open assumptions in the set  $\Gamma$ . But, by the induction hypothesis, there is a proof of  $B$  from open assumptions in the set  $\Gamma \cup \{A\}$  of the form:

$$\begin{array}{c} \Gamma \quad A \\ \vdots \\ B \end{array}$$

But then we may terminate the proof with an application of the  $\Rightarrow_I$  rule, as in:

$$\frac{\begin{array}{c} \Gamma \quad A \\ \vdots \\ B \end{array}}{A \Rightarrow B} \Rightarrow_I$$

yielding a proof of the desired form.

- Suppose the last rule applied is:

$$\frac{\Gamma \longrightarrow A}{\Gamma, \neg A \longrightarrow C} \neg_L$$

Then we need to show that there is a Natural Deduction proof of  $C$  from open assumptions in the set  $\Gamma, \neg A$ . But, by the induction hypothesis, there is a proof of  $A$  from open assumptions in the set  $\Gamma$  of the form:

$$\begin{array}{c} \Gamma \\ \vdots \\ A \end{array}$$

But then we may terminate the proof with an application of the  $\neg_E$  and  $\perp_E$  rules, as in:

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ A \end{array} \quad \neg A}{\perp} \neg_E \quad \frac{\perp}{C} \perp_E$$

yielding a proof of the desired form.

- Suppose the last rule applied is:

$$\frac{\Gamma, A \longrightarrow \perp}{\Gamma \longrightarrow \neg A} \neg_R$$

Then we need to show that there is a Natural Deduction proof of  $\neg A$  from open assumptions in the set  $\Gamma$ . But, by the induction hypothesis, there is a proof of  $\perp$  from open assumptions in the set  $\Gamma \cup \{A\}$  of the form:

$$\begin{array}{c} \Gamma \quad A \\ \vdots \\ \perp \end{array}$$

But then we may terminate the proof with an application of the  $\neg_I$  rule, as in:

$$\frac{\begin{array}{c} \Gamma \quad A \\ \vdots \\ \perp \end{array}}{\neg A} \neg_I$$

yielding a proof of the desired form.

Q.E.D.