

Orders: Partial, Total and Strict

Definition: A reflexive, transitive, antisymmetric relation on a set A is called a *partial order* on A , often denoted by a symbol such as \leq .

Definition: If R is a partial order on A , and for every x and y in A either $R(x, y)$ or $R(y, x)$, then R is called a *total order*.

Definition: A transitive, antireflexive, antisymmetric relation on a set A is called a *strict order* on A , often denoted by a symbol such as $<$.

If I_A is the identity relation consisting of (x, x) for all x in A , then for any partial order \leq over A , a corresponding strict order $<$ can be constructed as $\leq - I_A$.

Definition: Given a partial order \leq on a set A , the pair (A, \leq) is called a *partially ordered set*, or a *poset*.

Upper and Lower Bounds

Definition: Given a poset (A, \leq) , and given any subset X of A , an element $b \in A$ is a *lower bound* of X iff for all $x \in X, b \leq x$. An element $m \in A$ is an *upper bound* of X iff for all $x \in X, x \leq m$.

Note that b or m need not belong to X , and need not be unique. However, an upper or lower bound of X in X is unique.

Definition: A lower or upper bound of X in X is called *the least element* (or *the greatest element*) of X .

Such elements need not exist, but if they do, they are unique.

Minimal and Maximal Elements

Definition: Given a poset (A, \leq) , and given any subset X of A , an element $b \in X$ is a *minimal element of X* iff:

Definition: Given a poset (A, \leq) , and given any subset X of A , an element $b \in X$ is a *maximal element of X* iff:

Unlike least and greatest elements, minimal and maximal elements (if they exist) are not necessarily unique.

LUBs and GLBs

Definition: Given a poset (A, \leq) , and given any subset X of A , an element $m \in A$ is *the least upper bound of X* iff the set of upper bounds of X is nonempty, and m is the least element of this set.

Definition: Given a poset (A, \leq) , and given any subset X of A , an element $b \in A$ is *the greatest lower bound of X* iff the set of lower bounds of X is nonempty, and b is the greatest element of this set.

Can a set have an upper bound (resp. lower bound) and not have a least upper bound (resp. greatest lower bound)?

Sequences and Well-Founded Sets

Definition: Given two sets I and X , an *I -indexed sequence of elements of X* is any function $A : I \rightarrow X$, usually denoted by $(A_i)_{i \in I}$. The set I is called the *index set*.

Definition: A poset (A, \leq) , is *well-founded* iff it has no infinite decreasing sequence $(x_i)_{i \in \mathcal{N}}$. That is, if $<$ is the strict order associated with \leq , there is no sequence $(x_i)_{i \in \mathcal{N}}$ such that $x_{i+1} < x_i$ for all $i \geq 0$.

Well-Founded Sets

Lemma: Given a poset (A, \leq) , and a subset X of A , if X has no minimal element, then for every $x \in X$, there is a $y \in X$ such that $y < x$ (where $<$ is the strict order associated with \leq).

Proof:

Well-Founded Sets

Lemma: Given a poset (A, \leq) , (A, \leq) is a well-founded set iff every non-empty subset of A has a minimal element.

Proof: