

## Logical Consequence

The flip-side of Theorem 2.5.12 is a bit surprising at first:

**Theorem:** For any formula  $\Phi \in PROP$ ,  $\perp \models \Phi$ .

**Proof.**

## Logical Consequence

**Definition:** Given two formulas  $\Phi, \Psi \in PROP$ , we say that  $\Phi$  is *logically stronger than*  $\Psi$  if  $\Phi \models \Psi$  but  $\Psi \not\models \Phi$ . Or, equivalently, if  $\models \Phi \Rightarrow \Psi$  but  $\not\models \Psi \Rightarrow \Phi$

## Theories

**Definition:** Given a set of formulas  $U = \{\Phi_1, \dots, \Phi_n\} \subseteq PROP$ , the set of formulas  $\mathcal{T}(U) = \{\Psi \mid U \models \Psi\}$  is called *the theory of  $U$* . Each element of  $\mathcal{T}(U)$  is called a *theorem*.

## Logical Equivalence

As with  $\Rightarrow$  and  $\models$ , there is a meta-level partner to the operator  $\equiv$ .

**Definition:** Given two formulas  $\Phi, \Psi \in PROP$ , if for all valuations  $v$ ,  $v'(\Phi) = v'(\Psi)$ , we say that  $\Phi$  is *logically equivalent to*  $\Psi$ , written  $\Phi \leftrightarrow \Psi$ .

**Theorem (2.44):** Given two formulas,  $\Phi, \Psi \in PROP$ ,  $\Phi \leftrightarrow \Psi$  iff  $\models (\Phi \equiv \Psi)$ .

**Corollary:** For two formulas,  $\Phi, \Psi \in PROP$ ,  $\Phi \leftrightarrow \Psi$  iff:

- $\models (\Phi \Rightarrow \Psi)$  and  $\models (\Psi \Rightarrow \Phi)$
- $\models ((\Phi \Rightarrow \Psi) \wedge (\Psi \Rightarrow \Phi))$
- $\Phi \models \Psi$  and  $\Psi \models \Phi$

## Substitution

**Definition:** Given  $\Phi, \Psi \in PROP$  such that  $\Phi$  is a subformula of  $\Psi$ , then, for any formula  $\Xi \in PROP$ , the *substitution of  $\Xi$  for  $\Phi$  in  $\Psi$* , written  $\Psi[\Phi \leftarrow \Xi]$ , is the formula whose formation tree is obtained by replacing the formation subtree of  $\Phi$  in the formation tree of  $\Psi$  with the formation tree of  $\Xi$ .

This slightly cleaned up version of the definition given in the book is still not well defined. Why?

## The Substitution Theorem

**Theorem (2.4.7):** If formulas  $\Phi, \Psi, \Xi \in PROP$  are such that  $\Phi$  is a subformula of  $\Psi$ , and  $\Xi \leftrightarrow \Phi$ , then  $\Psi \leftrightarrow \Psi[\Phi \leftarrow \Xi]$ .

**Proof.**

## Some Useful Equivalences

Laws of 0 and 1:	$A \vee \top \leftrightarrow \top$ $A \vee \perp \leftrightarrow A$ $A \Rightarrow \top \leftrightarrow \top$ $A \Rightarrow \perp \leftrightarrow \neg A$	$A \wedge \top \leftrightarrow A$ $A \wedge \perp \leftrightarrow \perp$ $\top \Rightarrow A \leftrightarrow A$ $\perp \Rightarrow A \leftrightarrow \top$
More 0 and 1:	$A \vee \neg A \leftrightarrow \top$ $\top \leftrightarrow A \Rightarrow A$ $\top \leftrightarrow A \equiv A$ $\neg A \leftrightarrow A \uparrow A$	$A \wedge \neg A \leftrightarrow \perp$ $\perp \leftrightarrow A \not\equiv A$ $\neg A \leftrightarrow A \downarrow A$
Double Negation:	$A \leftrightarrow \neg\neg A$	
Idempotency:	$A \leftrightarrow A \wedge A$	$A \leftrightarrow A \vee A$
Commutativity:	$A \vee B \leftrightarrow B \vee A$ $A \equiv B \leftrightarrow B \equiv A$ $A \uparrow B \leftrightarrow B \uparrow A$	$A \wedge B \leftrightarrow B \wedge A$ $A \not\equiv B \leftrightarrow B \not\equiv A$ $A \downarrow B \leftrightarrow B \downarrow A$
Contrapositive:	$A \Rightarrow B \leftrightarrow \neg B \Rightarrow \neg A$	
Associativity:	$A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C$ $A \equiv (B \equiv C) \leftrightarrow (A \equiv B) \equiv C$	$A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$ $A \not\equiv (B \not\equiv C) \leftrightarrow (A \not\equiv B) \not\equiv C$
Distributivity:	$A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$	$A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)$
Absorption:	$A \wedge (A \vee B) \leftrightarrow A$	$A \vee (A \wedge B) \leftrightarrow A$
DeMorgan's Laws:	$A \Rightarrow B \leftrightarrow \neg A \vee B$ $A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)$ $A \vee B \leftrightarrow \neg A \Rightarrow B$	$A \Rightarrow B \leftrightarrow \neg(A \wedge \neg B)$ $A \wedge B \leftrightarrow \neg(\neg A \vee \neg B)$ $A \wedge B \leftrightarrow \neg(A \Rightarrow \neg B)$
Other Defs:	$A \equiv B \leftrightarrow (A \Rightarrow B) \wedge (B \Rightarrow A)$ $A \Rightarrow B \leftrightarrow A \equiv (A \wedge B)$ $A \wedge B \leftrightarrow (A \equiv B) \equiv (A \vee B)$	$A \Rightarrow B \leftrightarrow B \equiv (A \vee B)$ $A \equiv B \leftrightarrow (A \vee B) \Rightarrow (A \wedge B)$