Logically Complete Sets of Connectives

Definition: A set of connectives is *logically complete* if all the other connectives can be defined in terms of connectives from the set.

In other words, Given a set $C \subset CONN$, C is logically complete if for all propositions $\Phi \in PROP$, there is a proposition $\Phi_C \in PROP$ such that Φ_C uses only connectives from C, and $\Phi \leftrightarrow \Phi_C$.

Logically Complete Sets of Connectives

Theorem: The following are logically complete sets of connectives:

- $\{\neg, \vee\}$
- $\{\neg, \wedge\}$
- $\{\neg, \Rightarrow\}$
- $\{\uparrow\}$
- $\{\downarrow\}$

Proof.

Logically Complete Sets of Connectives

Theorem (2.4.8)**:** The sets $\{\uparrow\}$ and $\{\downarrow\}$ are the only logically complete singleton sets.

Proof.

Decision Procedures

How do we know if a formula $\Phi \in PROP$ is valid?

We need a *decision procedure*. I.e. an algorithm that, given a formula, returns **true** if the formula is valid and **false** if it is not (that is, if it is falsifiable).

Is there such an algorithm that is guaranteed to terminate?

Sure…

Further, Theorem 2.5.11 enables us to extend this procedure to determine if a formula is a logical consequence of a set of formulas.

But there are real problems with this method…

Deduction Systems

A *deduction system* is a syntactic system for manipulating formulas designed to lead to conclusions which correspond to logical consequences.

Formulas are manipulated by way of *deduction rules* intended to mimic *sound* inferences. Each rule has *premises* and a *conclusion*. There are also *axioms* which behave like rules with no premises.

A well-formed deductive structure is called a *proof*, and has some *conclusion*.

A formula, Φ, which may occur as the conclusion of a proof is said to be *provable* (or *derivable*), written $\vdash \Phi$. A proof may contain some formulas which are used as *assumptions*. If there is a proof of Φ using assumptions from the set Γ, we write $\Gamma \vdash \Phi$ (Φ is *derivable from* Γ).

Note: If we are discussing more than one deduction system, then for clarity, if Φ can be derived from Γ using the rules of the system $\mathcal{D}$ (Φ is *derivable from* Γ *under* $\mathcal{D}$), we will write $\Gamma \vdash_{\mathcal{D}} \Phi$.

Soundness and Completeness

The goal is to create deduction systems in which $\Gamma \vdash \Phi$ iff $\Gamma \models \Phi$.

Definition: Given a deduction system, $\mathcal{D}$, if $\Gamma \vdash_{\mathcal{D}} \Phi$ implies $\Gamma \models \Phi$, we say that $\mathcal{D}$ is *sound*.

Definition: Given a deduction system, $\mathcal{D}$, if $\Gamma \models \Phi$ implies $\Gamma \vdash_{\mathcal{D}} \Phi$, we say that $\mathcal{D}$ is *complete*.