Random Sidebar

What is the simplest tautology?

What is the next simplest tautology?

What is the simplest tautology that has no uses of logical constants (i.e. $\top$ or $\bot$)?

Hilbert Systems

Historically, one of the earliest proposed formal proof systems were what are now called *Hilbert Systems* or *axiomatic systems*. Such systems are signified by having only one core deduction rule, *modus ponens*, but a number of structural axioms categorizing the behavior of the principal operator (usually implication).

The Hilbert system, $\mathcal{H}$, presented in the text uses the standard rule of *modus ponens*:

$$\frac{A \quad A \Rightarrow B}{B}\,_{MP}$$

as well as three axioms of implication:

1. $A \Rightarrow (B \Rightarrow A)$
2. $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
3. $(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$

Note that, to be precise, these are really *rule schema* and *axiom schema*. Just as Φ is not a formula, but, rather, meta-syntactic placeholders for formulas, These are not actual rules and axioms, but templates for them.

Hilbert Systems

Here is an example of a proof in this Hilbert System. We prove the simple tautology $p \Rightarrow p$.

$$\dfrac{\dfrac{}{p \Rightarrow (p \Rightarrow p)}\;1 \qquad \dfrac{\dfrac{}{p \Rightarrow ((p \Rightarrow p) \Rightarrow p)}\;1 \qquad \dfrac{}{(\overset{A}{p} \Rightarrow ((\overset{B}{p \Rightarrow p}) \Rightarrow \overset{C}{p}) \Rightarrow ((\overset{A}{p} \Rightarrow (\overset{B}{p \Rightarrow p})) \Rightarrow (\overset{A}{p} \Rightarrow \overset{C}{p}))}\;2}{(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)}\;MP}{p \Rightarrow p}\;MP$$

Hilbert Systems

Suppose we have the following propositions:

- hj = "The jar is heated"
- bij = "The bug is in the jar"
- bd = "The bug is dead"

and we know that "If the jar is heated then if the bug is in the jar then the bug is dead." Suppose we know the stated rule, and that the jar is heated and that the bug is in the jar. Can we prove that the bug is dead?

The Deduction Rule / Theorem

While short proofs like the last are possible, The need to use the axioms to manipulate the structure of formulas means proofs are more often cumbersome. The solution was to introduce a series of *meta rules* which were proved to be sound and therefore *admissible*. These meta-rules allowed a variety of abbreviations in proofs.

The first and most important meta-rule is the *deduction rule*. It arises from the *Deduction Theorem*:

Theorem (2.9.5)**:** If $\Gamma \cup \{\Psi\} \vdash_{\mathcal{H}} \Phi$ then $\Gamma \vdash_{\mathcal{H}} \Psi \Rightarrow \Phi$.

Pictorially, this says that if there is a proof of the formula Φ of the form:

$$\begin{array}{c} \Gamma \cup \{\Psi\} \\ \vdots \\ \Phi \end{array}$$

Then there is a proof of the formula $\Psi \Rightarrow \Phi$ of the form:

$$\begin{array}{c} \Gamma \\ \vdots \\ \Psi \Rightarrow \Phi \end{array}$$

The Deduction Rule

Given the deduction theorem, which we will prove below, we can create a derived rule called the *Deduction Rule* that *discharges* one of the open assumptions of the proof of its premise.

We write the schema for the deduction rule as:

$$\frac{\begin{array}{c} \not\Psi \\ \vdots \\ \Phi \end{array}}{\Psi \Rightarrow \Phi}$$

Proof of The Deduction Theorem

Proof.

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