The Deduction Theorem

Theorem (2.9.5)**:** If $\Gamma \cup \{\Psi\} \vdash_{\mathcal{H}} \Phi$ then $\Gamma \vdash_{\mathcal{H}} \Psi \Rightarrow \Phi$.

Pictorially, this says that if there is a proof of the formula Φ of the form:

$$\begin{array}{c} \Gamma \cup \{\Psi\} \\ \vdots \\ \Phi \end{array}$$

Then there is a proof of the formula $\Psi \Rightarrow \Phi$ of the form:

$$\begin{array}{c} \Gamma \\ \vdots \\ \Psi \Rightarrow \Phi \end{array}$$

Proof of The Deduction Theorem

Proof. The (meta) proof follows a pattern that occurs frequently in logic, and follows by complete induction on the size (i.e. the height) of the proof tree. The proof breaks down into two cases, depending on whether the proof is of height one or greater than one.

Suppose the proof, Π, of the form

$$\begin{array}{c} \Gamma \cup \{\Psi\} \\ \vdots \\ \Phi \end{array}$$

is of height one ($h(\Pi) = 1$). Then the induction hypothesis is vacuous (since there are no proofs of lesser height) and either:

Proof of The Deduction Theorem

- Φ is an instance of axiom n. In which case the proof is actually just of the form:

$$\frac{}{\Phi}\,n$$

and the set of assumptions of the proof is actually empty, and there is no assumption Ψ, and this is not really an application of this theorem. Nonetheless, for *any* formula Ψ, the following is a valid proof tree:

$$\frac{\frac{}{\Phi}\,n \quad \frac{}{\Phi \Rightarrow (\Psi \Rightarrow \Phi)}\,1}{\Psi \Rightarrow \Phi}\,MP$$

- $\Psi = \Phi$, and the proof is of the form:

$$\Phi$$

with Φ as both conclusion and an open assumption. But we know that for any formula Φ there is a proof with the formula $\Phi \Rightarrow \Phi$ as its conclusion with the same structure as the proof of $(p \Rightarrow p)$ given earlier:

$$\frac{\frac{}{\Phi \Rightarrow (\Phi \Rightarrow \Phi)}\,1 \quad \frac{\frac{}{\Phi \Rightarrow ((\Phi \Rightarrow \Phi) \Rightarrow \Phi)}\,1 \quad \frac{}{(\Phi \Rightarrow ((\Phi \Rightarrow \Phi) \Rightarrow \Phi)) \Rightarrow ((\Phi \Rightarrow (\Phi \Rightarrow \Phi)) \Rightarrow (\Phi \Rightarrow \Phi))}\,2}{(\Phi \Rightarrow (\Phi \Rightarrow \Phi)) \Rightarrow (\Phi \Rightarrow \Phi)}\,MP}{\Phi \Rightarrow \Phi}\,MP$$

Proof of The Deduction Theorem

Now, suppose $h(\Pi) > 1$, and that for all proofs Π' such that $h(\Pi') < h(\Pi)$ the deduction theorem holds. Then the last step of the proof must be an application of *modus ponens*, and the proof, Π, is of the form:

$$\dfrac{\begin{array}{c}\Gamma \cup \{\Psi\} \\ \vdots \\ \Xi\end{array} \qquad \begin{array}{c}\Gamma \cup \{\Psi\} \\ \vdots \\ \Xi \Rightarrow \Phi\end{array}}{\Phi}\, MP$$

for some formula Ξ.

But then the proof of each premise has height less than $h(\Pi)$ and by the induction hypothesis the deduction theorem holds for those proofs. Therefore, there exist proofs of the form:

$$\begin{array}{c}\Gamma \\ \vdots \\ \Psi \Rightarrow \Xi\end{array} \qquad \text{and} \qquad \begin{array}{c}\Gamma \\ \vdots \\ \Psi \Rightarrow (\Xi \Rightarrow \Phi)\end{array}$$

But then the following structure is a valid proof:

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \Psi \Rightarrow \Xi\end{array} \qquad \dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \Psi \Rightarrow (\Xi \Rightarrow \Phi)\end{array} \qquad \dfrac{}{(\Psi \Rightarrow (\Xi \Rightarrow \Phi)) \Rightarrow ((\Psi \Rightarrow \Xi) \Rightarrow (\Psi \Rightarrow \Phi))}\, 2}{(\Psi \Rightarrow \Xi) \Rightarrow (\Psi \Rightarrow \Phi)}\, MP}{\Psi \Rightarrow \Phi}\, MP$$

∎

Impact of The Deduction Theorem

Thus, the deduction theorem holds, and the deduction rule is admissible.

This can significantly shorten proofs. For example, the previous proof of $(p \Rightarrow p)$:

$$\dfrac{\dfrac{}{p \Rightarrow (p \Rightarrow p)}\,1 \quad \dfrac{\dfrac{}{p \Rightarrow ((p \Rightarrow p) \Rightarrow p)}\,1 \quad \dfrac{}{(p \Rightarrow ((p \Rightarrow p) \Rightarrow p)) \Rightarrow ((p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p))}\,2}{(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)}\,MP}{p \Rightarrow p}\,MP$$

may now be replaced by the following proof using this derived rule:

$$\dfrac{\not p}{p \Rightarrow p}\,DR$$