The Deduction Rule / Theorem

Recall that the Deduction Theorem says that if $\Gamma \cup \{\Psi\} \vdash_{\mathcal{H}} \Phi$ then $\Gamma \vdash_{\mathcal{H}} \Psi \Rightarrow \Phi$.

That is, if there is a proof of the formula Φ of the form:

$$\begin{array}{c} \Gamma \cup \{\Psi\} \\ \vdots \\ \Phi \end{array}$$

then there is a proof of the formula $\Psi \Rightarrow \Phi$ of the form:

$$\begin{array}{c} \Gamma \\ \vdots \\ \Psi \Rightarrow \Phi \end{array}$$

This allows us to define a new system $\mathcal{H}'$ which is the same as $\mathcal{H}$ but with the addition of the Deduction Rule:

$$\frac{\begin{array}{c} \cancel{\Psi} \\ \vdots \\ \Phi \end{array}}{\Psi \Rightarrow \Phi}$$

Additional Derived Rules

It is possible to augment the Hilbert System $\mathcal{H}'$ with additional derived rules. For each rule we must prove that the rule is sound by showing that if we can prove something in the system that has the rule, we can still prove it in the system that doesn't have the rule (though the proof in that system may be much more complex).

Each addition technically leads to a new system, but we will refer to all of them as $\mathcal{H}'$, and use $\mathcal{H}$ to refer to the original pure system (with one rule and three axioms).

The Contrapositive Rule

$$\frac{(\neg B) \Rightarrow (\neg A)}{A \Rightarrow B}$$

The Transitivitry Rule

$$\frac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}$$

Exchange of Antecedent

$$\frac{A \Rightarrow (B \Rightarrow C)}{B \Rightarrow (A \Rightarrow C)}$$

The Double Negation Rule

$$\frac{\neg\neg A}{A}$$

The Or Rule

$$\frac{A}{A \vee B}$$