Natural Deduction

In 1935, Gerhard Gentzen, proposes a new way of formulating logic.

In *Investigations Into Logical Deduction* he lays out what amounts to a particular collection of derived rules, but then shows that these rules, without any axioms, are themselves complete.

These rules also correspond to a more natural way of writing proofs, and he therefore calls the system *Natural Deduction*.

Natural Deduction

There are two rules for each operator, an *Introduction Rule*, and an *Elimination Rule*.

Introduction Rules

$$\dfrac{\begin{array}{c}\not\psi\\ \vdots\\ \phi\end{array}}{\psi \Rightarrow \phi} \Rightarrow_I$$

$$\dfrac{\phi \quad \psi}{\phi \wedge \psi} \wedge_I$$

$$\dfrac{\phi}{\phi \vee \psi} \vee_{I_1} \qquad \dfrac{\psi}{\phi \vee \psi} \vee_{I_2}$$

$$\dfrac{\begin{array}{cc}\not\psi & \not\phi\\ \vdots & \vdots\\ \phi & \psi\end{array}}{\psi \equiv \phi} \equiv_I$$

$$\dfrac{\begin{array}{c}\not\phi\\ \vdots\\ \bot\end{array}}{\neg\phi} \neg_I$$

Elimination Rules

$$\dfrac{\psi \quad \psi \Rightarrow \phi}{\phi} \Rightarrow_E$$

$$\dfrac{\phi \wedge \psi}{\phi} \wedge_{E_1} \qquad \dfrac{\phi \wedge \psi}{\psi} \wedge_{E_2}$$

$$\dfrac{\phi \vee \psi \quad \begin{array}{c}\not\phi\\ \vdots\\ \xi\end{array} \quad \begin{array}{c}\not\psi\\ \vdots\\ \xi\end{array}}{\xi} \vee_E$$

$$\dfrac{\psi \quad \psi \equiv \phi}{\phi} \equiv_E \qquad \dfrac{\phi \quad \psi \equiv \phi}{\psi} \equiv_E$$

$$\dfrac{\phi \quad \neg\phi}{\bot} \neg_E$$

$$\dfrac{\bot}{\phi} \bot_E$$

Classical Natural Deduction

As shown so far, the system is *constructive*.

In order to be able to do proofs by contradiction, which do not produce a witness, and thereby to make the system *classical*, you can either:

- Allow axioms of the form $\phi \vee \neg\phi$ (explicit invocations of the law of the excluded middle).
- Or, add the double negation rule to the system:

$$\frac{\neg\neg\phi}{\phi}$$

- Or, add the rule *reductio ad absurdum*:

$$\begin{array}{c} \not{\neg\phi} \\ \vdots \\ \dfrac{\bot}{\phi}\ RAA \end{array}$$

This ability to distinguish intuitionistic from classical reasoning based just on the availability of certain rules, further distinguishes natural deduction from Hilbert systems, in which it is harder to see how to make the separation.

Classical Natural Deduction

Note that while it is quite similar, this last, RAA, is not the same as the $\neg_E$ rule:

$$\dfrac{\begin{array}{c}\not\phi \\ \vdots \\ \bot\end{array}}{\neg\phi}\ \neg E \qquad\qquad \dfrac{\begin{array}{c}\neg\not\phi \\ \vdots \\ \bot\end{array}}{\phi}\ RAA$$

What are the readings of the two rules?

Some Natural Deduction Proofs

$$\frac{\vdots}{p \Rightarrow p}$$

$$\frac{\vdots}{(a \wedge b) \Rightarrow (b \wedge a)}$$

$$\frac{\vdots}{(a \vee b) \Rightarrow (b \vee a)}$$

Some Natural Deduction Proofs

$$\frac{\vdots}{\phi \vee \neg\phi}$$