

Gentzen Sequent Calculus

A **Sequent** is a syntactic structure of the form:

$$\Gamma \longrightarrow \Delta$$

where Γ, Δ are (possibly empty) comma-delimited sets of well-formed formulas.

You can read an individual sequent as a meta-statement about a logical consequence:

If all of the formulas in Γ are true, then at least one of the formulas in Δ is true.

Note that this is not the same as saying:

At least one of the formulas in Δ is a logical consequence of Γ .

Why not?

We will begin by looking at the rules for the intuitionistic calculus, in which the right-hand-side, Δ is restricted to being at most one formula.

LJ – Intuitionistic Gentzen Sequent Calculus

Assume Γ, Δ are (possibly empty) sets of wffs, and A, B, C are wffs.

The rules can be read from the top down (giving instructions on how to construct a sound deduction) or bottom up (giving instructions on how to search for a proof):

$$\overline{\Gamma, A \longrightarrow A} \textit{id} \quad \overline{\Gamma \longrightarrow \top} \top \quad \overline{\Gamma, \perp \longrightarrow A} \perp$$

$$\frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \wedge_L \quad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge_R$$

$$\frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C} \vee_L \quad \frac{\Gamma \longrightarrow A_i}{\Gamma \longrightarrow A_1 \vee A_2} \vee_R$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \Rightarrow B \longrightarrow C} \Rightarrow_L \quad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \Rightarrow B} \Rightarrow_R$$

$$\frac{\Gamma \longrightarrow A}{\Gamma, \neg A \longrightarrow C} \neg_L \quad \frac{\Gamma, A \longrightarrow \perp}{\Gamma \longrightarrow \neg A} \neg_R$$

LK – Classical Gentzen Sequent Calculus

To reason classically, we simply expand the right-hand-side of the sequent to a set of formulas:

$$\overline{\Gamma, A \longrightarrow \Delta, A} \textit{id} \quad \overline{\Gamma \longrightarrow \Delta, \top} \top \quad \overline{\Gamma, \perp \longrightarrow \Delta} \perp$$

$$\frac{\Gamma, A, B \longrightarrow \Delta}{\Gamma, A \wedge B \longrightarrow \Delta} \wedge_L \quad \frac{\Gamma \longrightarrow \Delta, A \quad \Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, A \wedge B} \wedge_R$$

$$\frac{\Gamma, A \longrightarrow \Delta \quad \Gamma, B \longrightarrow \Delta}{\Gamma, A \vee B \longrightarrow \Delta} \vee_L \quad \frac{\Gamma \longrightarrow \Delta, A, B}{\Gamma \longrightarrow \Delta, A \vee B} \vee_R$$

$$\frac{\Gamma \longrightarrow \Delta, A \quad \Gamma, B \longrightarrow \Delta}{\Gamma, A \Rightarrow B \longrightarrow \Delta} \Rightarrow_L \quad \frac{\Gamma, A \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, A \Rightarrow B} \Rightarrow_R$$

$$\frac{\Gamma \longrightarrow \Delta, A}{\Gamma, \neg A \longrightarrow \Delta} \neg_L \quad \frac{\Gamma, A \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \neg A} \neg_R$$

$$\frac{\Gamma \longrightarrow \Delta, A, B \quad \Gamma, A, B \longrightarrow \Delta}{\Gamma, A \equiv B \longrightarrow \Delta} \equiv_L \quad \frac{\Gamma, A \longrightarrow \Delta, B \quad \Gamma, B \longrightarrow \Delta, A}{\Gamma \longrightarrow \Delta, A \equiv B} \equiv_R$$

Some Sequent Calculus Proofs

$$\frac{}{\longrightarrow p \Rightarrow p}$$

$$\frac{}{\longrightarrow (a \wedge b) \equiv (b \wedge a)}$$

$$\frac{}{\longrightarrow (a \vee b) \equiv (b \vee a)}$$

Some Sequent Calculus Proofs

$$\frac{\vdots}{\longrightarrow (hj \Rightarrow (bij \Rightarrow ((hj \Rightarrow (bij \Rightarrow bd)) \Rightarrow bd)))}$$