# Logic and Mathematics

To apply FOPC to mathematical reasoning, we must *axiomatize* the structures about which we wish to reason. That is, we write down a set of first-order sentences (formulas with no free variables), $\Gamma$, that describe the structure. Any model of $\Gamma$ is then such a structure.

To deduce properties of a structure, we ask what formulas are logical consequences of $\Gamma$, i.e., for what $A$ does $\Gamma \models A$.

Today we will just look at examples of writing down such axiomatizations.

One question of interest to logicians is whether a system has a *finite axiomatization*, or not.

It is also interesting to discover that there are mathematical properties of structures that can be determined just by looking at the meta-structure of their axiomatization.

# Partial Orders

**Definition:** A structure is a *partially ordered set* if it is a set together with a reflexive, transitive, anti-symmetric relation.

**Language:**

$\mathcal{A} = \{ \qquad \qquad \}$ $\quad \mathcal{F} = \{ \qquad \qquad \}$ $\quad \mathcal{P} = \{ \qquad \qquad \}$

**$\Gamma$:**

**Definition:** A poset is *totally* or *linearly ordered* if all elements are comparable.

**$\Gamma$:**

**Definition:** A poset is *densely ordered* if between any two distinct elements there is a third one.

**$\Gamma$:**

## Partial Orders

**Definition:** A poset has an *upper bound* if there is an element at least as big as any other element.

$\Gamma$**:**

**Definition:** A poset has a *least upper bound* if it has an upper bound that is as small as any upper bound.

$\Gamma$**:**

Note that any property that requires refering to subsets as individuals is second-order, meaning it is not axiomatizable as a set of first-order sentences.

# Groups

**Definition:** A structure is a *group* if it has an associative function, an identity element for that function (left and right), and a unary inverse function such that when the principal function is applied to an element and its inverse, the result is the identity element.

**Language:**

$\mathcal{A} = \{ \qquad \}$ $\quad \mathcal{F} = \{ \qquad \}$ $\quad \mathcal{P} = \{ \qquad \}$

**$\Gamma$:**

**Definition:** A group is *abelian* or *commutative* if the principal function is commutative.

**$\Gamma$:**

# Rings with Unity

**Definition:** A structure is a *ring* (with unity) if it has two associative functions, ($+$ and $\cdot$) each with an identity element (different from the other). The first function is commutative and there is a unary inverse function for it. (I.e., if we consider the first function alone, the structure is an abelian group.) Finally, the second function distributes over the first.

**Language:**

$\mathcal{A} = \{ \qquad \}$ $\quad \mathcal{F} = \{ \qquad \}$ $\quad \mathcal{P} = \{ \qquad \}$

**$\Gamma$:**

# Rings with Unity

**Definition:** A ring is *commutative* if the second function is also commutative.

**Γ:**

**Definition:** A ring is a *division ring* if each element other than the identity of the first function has an inverse under the second function.

**Γ:**

**Definition:** A commutative division ring is called a *field*.