First-Order Gentzen Sequent Calculus

As with Natural Deduction, there are four new rules for the quantifiers. In this case, though, they are left and right rules rather than introduction and elimination rules:

$$\frac{\Gamma, \Phi[x := t] \longrightarrow \Delta}{\Gamma, \forall x(\Phi) \longrightarrow \Delta}\,\forall_L \qquad \frac{\Gamma \longrightarrow \Delta, \Phi[x := c]}{\Gamma \longrightarrow \Delta, \forall x(\Phi)}\,\forall_{R*}$$

$$\frac{\Gamma, \Phi[x := c] \longrightarrow \Delta}{\Gamma, \exists x(\Phi) \longrightarrow \Delta}\,\exists_{L*} \qquad \frac{\Gamma \longrightarrow \Delta, \Phi[x := t]}{\Gamma \longrightarrow \Delta, \exists x(\Phi)}\,\exists_R$$

with the following proviso:

* - The constant c may not occur free in the lower sequent.

First-Order Gentzen Sequent Calculus

As before, to see the importance of the provisos, consider the following bad "proofs":

$$\frac{}{\longrightarrow (0 = 0) \Rightarrow \forall x(x = 0)}$$

$$\frac{}{\longrightarrow \forall x(\neg\forall y(x = y)) \Rightarrow (\neg\forall y(y = y))}$$

First-Order Gentzen Sequent Proofs

$$\overline{\longrightarrow \forall x(\forall y(\Phi(x,y))) \Rightarrow \forall y(\forall x(\Phi(x,y)))}$$

$$\overline{\longrightarrow \forall x(\Phi \wedge \Psi) \Rightarrow (\forall x(\Phi) \wedge \forall x(\Psi))}$$

First-Order Gentzen Sequent Proofs

$$\frac{}{\longrightarrow \exists x(\Phi \wedge \Psi) \Rightarrow (\exists x(\Phi) \wedge \exists x(\Psi))}$$

$$\frac{}{\longrightarrow (\exists x(\Phi) \wedge \exists x(\Psi) \Rightarrow \exists x(\Phi \wedge \Psi))}$$

First-Order Gentzen Sequent Proofs

$$\frac{}{\longrightarrow \forall x(\Phi(x)) \Rightarrow \exists y(\Phi(y))}$$

$$\frac{}{\longrightarrow \exists x(\Phi(x)) \Rightarrow \forall y(\Phi(y))}$$

Inference in First-Order Gentzen Sequent Calculus

- If a bug is in a heated jar, its dead.
- This jar is heated.
- This bug is in this jar.
- This bug is dead.