Skolemization

The technique for finding the corresponding clausal formula is based on the following observation:

Suppose the formula $\exists x(\Phi)$ is satisfiable.

What does that mean?

That there is some assignment to the variable x which makes Φ true.

But the same holds for the formula $\Phi[x := c]$, where c is some constant not otherwise appearing in Φ. It is satisfied if some interpretation maps c to a domain value such that $\Phi[x := c]$ is true.

So, if $\exists x(\Phi)$ is satisfiable, then $\Phi[x := c]$ is satisfied by any interpretation that interprets the constant c as the domain value that the assignment that satisfied $\exists x(\Phi)$ assigned to x.

The constant we substitue for the existentially quantified variable is called the *skolem constant*, and the process of constructing the clausal formula is called *skolemization*.

Skolemization

We must be careful, however. Consider the formula

$$\forall x(\exists y(p(x, y)))$$

It is satisfiable (for instance, in the domain of the integers where p is the predecessor relation), but there is an implicit dependence of the value for y that satisfies the formula on the value chosen for x. For each x there is a (potentially) different y that satisfies the formula.

Therefore, in skolemizing an existentially quantified variable, we replace it with a constant only if there are no universal quantifiers to the left of the existential in the prefix. Otherwise, if there are n universal quantifiers to the left of the existential, we replace the existentially quantified variable with a term consisting of a new *function symbol* of arity n, applied to the n universally quantified variables.

Thus, in the example above, the mutually-satisfiable clausal formula is:

$$\forall x(p(x, f(x))$$

Herbrand's Theorems

Herbrand, building on results of Lowenheim, Skolem, and Gödel, proved a series of theorems that intimately tie together satisfiability (and hence validity), which are semantic properties, with syntax. These theorems form the foundation of automated theorem proving.

It is not just that they provide a hint at the techniques to be used. In a deeper sense, if they were not true, then automated theorem proving simply would not be possible.

Herbrand Universes

Definition: (3.9.1) Let Γ be a set of clausal formulas. Let $\mathcal{A}_\Gamma$, and $\mathcal{F}_\Gamma$, be, respectively, the sets of constants and function symbols appearing in the formulas of Γ. Then the *Herbrand Universe* of Γ is the set $\mathcal{H}_\Gamma \subset TERM$ defined inductively as:

- If $a \in \mathcal{A}_\Gamma$, then $a \in \mathcal{H}_\Gamma$.
- If $f \in \mathcal{F}_\Gamma$, is an arity n function symbol, and $t_1, \ldots t_n \in \mathcal{H}_\Gamma$, then $f(t_1, \ldots, t_n) \in \mathcal{H}_\Gamma$.

(If $\mathcal{A}_\Gamma = \emptyset$, then an arbitrary constant symbol a is used to give a basis to the inductive definition.)

Thus the Herbrand universe of a set of formulas is just the set of all terms that can be built out of constant and function symbols occurring in those formulas.

Herbrand's theorems tell us that we can at least limit our search for instantiating values to at most those terms.

Herbrand Models

Definition: (3.9.2) A *ground term* (resp. *atom*, *literal*, or *clause*) is a term (resp. atom, literal, or clause) which has had elements from the Herbrand Universe substituted for its variables.

Definition: (3.9.3) A *Herbrand Interpretation* is an interpretation where the domain is a Herbrand universe and where the interpretation function interprets each constant and function symbol as 'itself.'

Definition: (3.9.5) A *Herbrand model* for a set of clauses Γ is a Herbrand interpretation which satisfies Γ.

Herbrand Bases

Definition: (3.9.4) Let Γ be a set of clausal formulas. Let $\mathcal{P}_\Gamma$ be the set of predicate symbols appearing in the formulas of Γ. Then the *Herbrand Base* of Γ is the set $\mathcal{B}_\Gamma \subset ATOM$ defined as:

- If $p \in \mathcal{P}_\Gamma$, is an arity n predicate symbol, and $t_1, \ldots t_n \in \mathcal{H}_\Gamma$, then $p(t_1, \ldots, t_n) \in \mathcal{B}_\Gamma$.

Just as a propositional valuation is identified by the assignment of truth values it makes to the propositional letters, a Herbrand Model is identified by its assignment of truth values to the elements of $\mathcal{B}_\Gamma$.