CS140: Algorithms

Fall 2000

Solutions to Assignment 10

1. Problem 27.2 on page 626 of the text.

   (a) Build the network described in the hint. Let each edge have capacity 1. Run the max flow algorithm. We may assume that the flow on each edge is integral so it is either 0 or 1. If (x_i, y_j) has positive flow then add edge (i, j) to the path cover P.

   We need to show that this algorithm produces a minimum path cover. We first show that P is a path cover. Since G is acyclic, it is sufficient to show that every vertex i in G has at most 1 out-edge in P. Since x_i has one in-edge with capacity 1, at most one out-edge of x_i has positive flow. Therefore vertex i has at most one out-edge in P.

   Next we show that the path cover has the smallest possible number of paths. Let n be the number of vertices in G, k be the number of paths in P, and m be the number of edges in P. Then $n = e + k$. Thus minimizing the number of paths in P is equivalent to maximizing the number of edges in P. Now notice that any feasible (integral) flow with value ℓ in the network corresponds to a path cover with exactly ℓ edges. Thus the maximum flow produces a path cover with a maximum number of edges.

   (b) The algorithm fails if G has cycles. In this case the edges of G produced by the algorithm may contain cycles and hence are not guaranteed to form a path cover.

2. Problem 27.3 on page 627 of the text.

   (a) Consider a cut where E_j is blue and some $I_k \in R_j$ is red. The capacity of this cut is the sum of the capacities of the edges that go from a red vertex to a blue one. The infinite capacity edge from I_k to E_j is such an edge so the capacity of the cut is infinite. Thus in any finite capacity cut in which E_j is blue, $I_k \in R_j$ must also be blue.

   (b) Let (R, B) be a finite capacity cut in the network. The capacity of this cut is

   $$\sum_{E_i \in R} p_i + \sum_{I_j \in B} c_j.$$

   Minimizing this expression over all finite capacity cuts is equivalent to maximizing

   $$T - (\sum_{E_i \in R} p_i + \sum_{I_j \in B} c_j)$$

over all finite capacity cuts, where T is an arbitrary constant. In particular, if we let $T = \sum_{i=1}^{m} p_i$ then the last expression is equivalent to maximizing

$$\sum_{E_i \in B} p_i - \sum_{I_j \in B} c_i$$

under the condition that if $E_i \in B$ then $I_j \in R_i$ is also in B. Note that this last expression is the expression for the total profit if we do the blue experiments and buy the needed instruments.

(c) Given a maximal flow we can find a minimum cut by building the residual graph G_R. Then color a vertex u blue if there is a path from u to t in G_R and red otherwise. (Run time analysis was not required.)