CS140: Algorithms

Fall 2000

Solutions to Assignment 12

1. Let $G = (V, E)$ be a directed acyclic graph with n vertices and m edges. Design an algorithm that lists the vertices of G in an order such that if $< u, v >$ is an edge of G then u precedes v in the list. To receive full credit your algorithm should run in time $O(n + m)$.

   Run DFS with timestamps then list the vertices in decreasing order of departure time.

2. Let $G = (V, E)$ be a directed graph with n vertices and m edges. The transitive closure of G, denoted G^T, is the graph with vertex set V and edge set

   $$E^T = \{<u, w> \mid \text{there is a path from } u \text{ to } w \text{ in } G\}.$$

   Let $d(u, w)$ denote the length of the shortest path from vertex u to vertex w in G. The diameter of G is defined to be

   $$max\{d(u, w)| \text{ there is a path from u to v in } G\}.$$

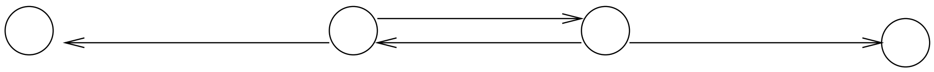
   An example of a graph and its transitive closure is shown in the figure on the next page. The diameter of the graph G in the figure is 3. What is the diameter of G^T?

   (a) Give an algorithm that takes as input the adjacency matrix M and diameter d of a directed graph G and outputs the adjacency matrix of G^T. To receive full credit your algorithm should run in time $O(d \cdot n^{\lg 7})$. You may assume that n is a power of 2. (Hint: Reduce the problem to a problem we've studied that has an algorithm with running time $O(n^{\lg 7})$.

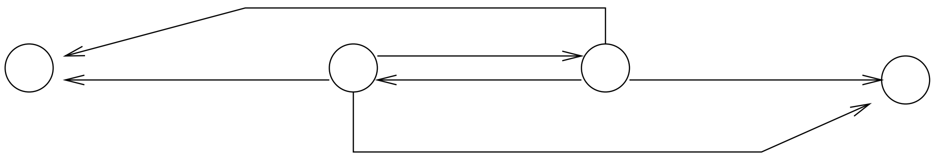
   If we raise M to the kth power then $M^k(i, j)$ is the number of paths of length k between vertices i and j. The adjacency matrix of G^T can be obtained by replacing each non-zero entry in the matrix $\sum_{i=1}^{d} M^i$ by a 1. We can use Strassen's matrix multiplication to compute $M, M^2, \ldots, M^d$ giving us an overall running time of $O(dn^{\lg 7}$.

   (b) Now suppose that G is acyclic. In this case we can assume, without loss of generality, that the adjacency matrix of G is upper triangular. Give an algorithm for finding the transitive closure of G in time $O(n^{\lg 7}))$. (Hint: Divide and conquer)

G:



G^T:



Let the vertices of G be numbered as described in problem 1. (Then the adjacency matrix for G is upper triangular.) Let G_1 (resp. G_2) be the subgraph of G induced by the first (resp. last) $n/2$ vertices. We'll recursive compute the transitive closure of G_1 and G_2. Now construct the graph G' consisting of G_1^T, G_2^T and all edges of G that go from a vertex in G_1 to a vertex in G_2. Note that the transitive closure of G' is also the transitive closure of G. Further G' has diameter of 3 or less. By part (a) above we can compute the transitive closure of G' in $O(n^{\lg 7})$. The overall running time of our algorithms is $T(n,m) = 2T(n/2, \max(m-1, (n/2)^2)) + cn^{\lg 7} = O(n^{\lg 7})$.

3. Run the Floyd-Warshall algorithm on the digraph whose adjacency matrix is given below. Show D^k for $k = 0, \ldots, 5$. (There are no self-loops, hence the 0's on the diagonal. An ∞ indicates a lack of edge.)

0	∞	∞	-1	20
∞	0	5	3	∞
∞	-2	0	1	∞
1	2	2	0	∞
3	3	2	∞	0

The adjacency matrix is D^0. D^1, D^2, D^3, D^4, and D^5 are given below. (Note: If you discover any mistakes in the following solution please let me know.)

D^1:

0	∞	∞	-1	20
∞	0	5	3	∞
∞	-2	0	1	∞
1	2	2	0	21
3	3	2	2	0

D^2:

0	∞	∞	-1	20
∞	0	5	3	∞
∞	-2	0	1	∞
1	2	2	0	21
3	3	2	2	0

D^3:

0	∞	∞	-1	20
∞	0	5	3	∞
∞	-2	0	1	∞
1	0	2	0	21
3	0	2	2	0

D^4:

0	1	0	-1	20
4	0	5	3	24
2	-2	0	1	22
1	0	2	0	21
3	0	2	2	0

D^5:

0	1	0	-1	20
4	0	5	3	24
2	-2	0	1	22
1	0	2	0	21
3	0	2	2	0