Algorithms
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Lecture 2
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Last class
The two important questions we consider in CS140:
– Is the algorithm correct?
– Is the algorithm fast?

Running Time
Where to measure?

A useful assumption
$T_A(Z)$: number of steps of algorithm on input $Z$
$T_M(Z)$: number of steps of machine on input $Z$
$T_A(Z)/c < T_M(Z) < cT_A(Z)$

Running Time:
What to measure?

Pick special case
• Run time depends on input size
• Run time can vary on different inputs of size $n$
Worst case performance of algorithm ▲

- We can compute this function at a finite number of points.
- Better yet, we can model this function for all input sizes.

A general problem …

- Question: How can we give a succinct description of an arbitrary function?
- Answer: Big-O notation.

Upper Bounds

- $f : \mathbb{N} \rightarrow \mathbb{N}$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ are positive-valued, monotonically increasing functions.
- $O(g(n)) = \{ f(n) : \text{there are constants } c \text{ and } M \text{ such that } f(n) \leq c \cdot g(n) \text{ for all } n \geq M \}$

Examples:

- Is $n^2 \in O(n^3)$?
- Is $2^n \in O(n^3)$?

We will also say $f(n) = O(g(n))$ to mean $f(n) \in O(g(n))$

CS140 pragmatism

What is the asymptotic behavior of the worst-case running time of the algorithm?
What is the asymptotic behavior of the worst-case running time of the algorithm?

Run time bounds for algorithm $\mathcal{A}$

- The running time of $\mathcal{A}$ is $O(n^3)$.
- The worst case running time of $\mathcal{A}$ is $O(n^3)$.
- $\mathcal{A}$ is $O(n^3)$.

Rate of growth of common functions

- Review of properties/notation
- See CLR pp 32-37 for details

**KNOW THIS STUFF**
Some useful observations about Big-O

- Transitivity: \( f(n) = O(g(n)) \) and \( g(n) = O(h(n)) \) \( \Rightarrow f(n) = O(h(n)) \)

- If \( \lim_{n \to \infty} f(n)/g(n) \) is constant then \( f(n) = O(g(n)) \).

- If \( \lim_{n \to \infty} f(n)/g(n) \) is unbounded then \( f(n) \neq O(g(n)) \).

Logarithms

Compare the rate of growth of the following functions:

- \( \log n \)
- \( \log n^2 \)
- \( \log 10000n \)

Polynomials

Compare the rate of growth of the following functions:

- \( n \)
- \( n^2 \)
- \( 1000n^2 + n \)

Polynomially bounded functions

\( f(n) \) is polynomially bounded if there is a constant \( k \) such that \( f(n) = O(n^k) \)

Exponentials

- Compare the rate of growth of the following functions:
  - \( 2^n \)
  - \( 3^n \)
  - \( 2^{n^2} \)
  - \( (2^n)^2 \)

Some rules of thumb

- Logs are slower growing than polynomials: \( \log(n) = O(n^k) \) for any \( k > 0 \)

- Polynomials are slower growing than exponentials: \( n^k = O(r^n) \) for any \( k > 0, r > 1 \)
Logs, Polys, and Exps

- Compare the rate of growth of the following functions:
  - \( \log n \)
  - \( n^3 \)
  - \( 2^n \)
- Which are polynomially bounded?

Other functions

- Factorial: \( n! = n(n-1)! \), \( 0! = 1 \)
- Tower of 2: \( T^*(n) = 2^{T^*(n-1)}, T^*(0) = 1 \)
- Iterated log: \( \log^*(n) = m \) such that \( T^*(m-1) < n \leq T^*(m) \)
- Ceiling: \( \lceil \lceil n \rceil \rceil = 2^m \) such that \( m-1 < \log n \leq m \)

Logs, polys, exps, and others

- Compare the rate of growth of the following functions:
  - \( \log n \), \( n^3 \), \( 2^n \), \( n! \), \( T^*(n) \), \( \log^*(n) \), \( \lceil \lceil n \rceil \rceil \)
- Which of these functions are polynomially bounded?

Beyond O

real numbers

- \( \leq \)
- \( \geq \)
- \( = \)
- \( < \)
- \( > \)

functions

- \( O \)
- \( \Omega \)
- \( \Theta \)
- \( o \)
- \( \omega \)

Lower Bounds

- If \( f : \mathbb{N} \rightarrow \mathbb{N} \) and \( g : \mathbb{N} \rightarrow \mathbb{N} \) are positive-valued, monotonically increasing functions.
- \( \Omega(g(n)) = \{ f(n) : \text{there are constants } c \text{ and } M \text{ such that } f(n) \geq c g(n) \text{ for all } n \geq M \} \)

We will also say

\( f(n) = \Omega(g(n)) \) to mean \( f(n) \in \Omega(g(n)) \)
Definition: $\Theta$

$f(n) = \Theta(g(n))$ if the following hold:
1. $f(n) = O(g(n))$, and
2. $f(n) = \Omega(g(n))$

Definition: little-o, little-$\omega$

- $f(n) = o(g(n))$ if $\lim_{n \to \infty} f(n)/g(n) = 0$
- $f(n) = \omega(g(n))$ if $\lim_{n \to \infty} f(n)/g(n) = \infty$

Logs, polys, exps, and others

Compare the following functions. Which of $O$, $\Omega$, $\Theta$, $o$, and $\omega$ apply?

$\log n$, $n^3$, $2^n$, $n!$, $T^*(n)$, $\log^*(n)$, $\lceil n \rceil$

A slight twist…

Is $f(2n) = O(f(n))$?
1. $f(n) = 1$: Is $2n = O(n)$?
2. $f(n) = 3n$: Is $6n = O(3n)$?
3. $f(n) = n^2$: Is $4n^2 = O(n^2)$?
4. $f(n) = 2^n$: Is $4^x = O(2^n)$?
5. $f(n) = n!$: Is $(2n)! = O(n!)$?