Computer Graphics

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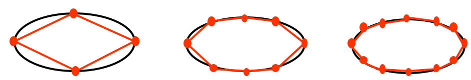
Lecture 12

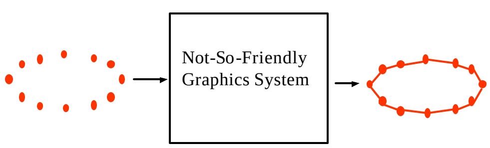
10/23/00

Modeling: Curves

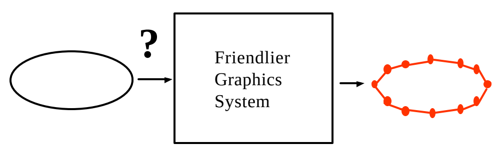
We can model curves to arbitrary precision with polylines:



Modeling Curves



Modeling Curves



Parametric Curve Representation

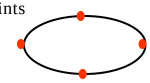
- $x=a_{x,0} + a_{x,1}t + \ldots + a_{x,n-1}t^{n-1} + a_{x,n}t^n$
- $y=a_{y,0} + a_{y,1}t + \ldots + a_{y,n-1}t^{n-1} + a_{y,n}t^n$

Linear: Line Segment

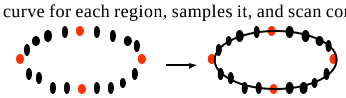
- The points (x(t),y(t),z(t)), $t \in [0,1]$, where
  $x(t)=a_{x,0} + a_{x,1}t$
  $y(t)=a_{y,0} + a_{y,1}t$
- User specifies the endpoints (x_0,y_0) and (x_1,y_1) then
  $a_{x,0} = x_0$, $a_{x,1} = x_1-x_0$
  $a_{y,0} = y_0$, $a_{y,1} = y_1-y_0$

Modeling: Curves

- The user specifies ``properties'' of the curve between sample points

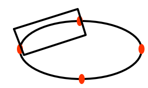


- The graphics system computes an appropriate parametric curve for each region, samples it, and scan converts



Modeling: Curves

- What degree curve should we use?
  - High degree $\Rightarrow$ Good model
  - High degree $\Rightarrow$ Slow computation
- User chooses adequate sample points



Cubic Curve

- The points (x(t),y(t)), $t \in [0,1]$, where

  $x(t)=a_{x,0} + a_{x,1}t + a_{x,2}t^2 + a_{x,3}t^3$

  $y(t)=a_{y,0} + a_{y,1}t + a_{y,2}t^2 + a_{y,3}t^3$

- User specifies endpoints (x(0),y(0)) and (x(1),y(1)) + ____________________

Cubic Curve: Interpolation

- The points (x(t),y(t)), $t \in [0,1]$, where

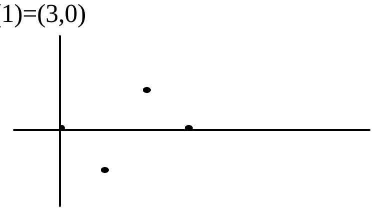
  $x(t)=a_{x,0} + a_{x,1}t + a_{x,2}t^2 + a_{x,3}t^3$

  $y(t)=a_{y,0} + a_{y,1}t + a_{y,2}t^2 + a_{y,3}t^3$

- User specifies endpoints (x(0),y(0)) and (x(1),y(1)) + two intermediate points, e.g. (x(1/3),y(1/3)) and (x(2/3),y(2/3))

Example

- p(0)=(0,0), p(1/3)=(1,-1), p(2/3)=(2,1), and p(1)=(3,0)



Example

- p(0)=(0,0), p(1/3)=(1,-1), p(2/3)=(2,1), and p(1)=(3,0)

- We want to find the coefficients of :

  $x(t) = a_{x,0} + a_{x,1}t + a_{x,2}t^2 + a_{x,3}t^3$

  given:

  $x(0)= \quad 0= a_{x,0}$

  $x(1/3)= 1= a_{x,0} + a_{x,1}(1/3) + a_{x,2}(1/9) + a_{x,3}(1/27)$

  $x(2/3)= 2= a_{x,0} + a_{x,2}(2/3) + a_{x,2}(4/9) + a_{x,3}(8/27)$

  $x(1)= \quad 3= a_{x,0} + a_{x,1} + a_{x,2} + a_{x,3}$

Matrix Form: x(t)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 1/9 & 1/27 \\ 1 & 2/3 & 4/9 & 8/27 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_{x,0} \\ a_{x,1} \\ a_{x,2} \\ a_{x,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution: x(t)

$$\begin{bmatrix} a_{x,0} \\ a_{x,1} \\ a_{x,2} \\ a_{x,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5.5 & 9 & -4.5 & 1 \\ 9 & -22.5 & 18 & -4.5 \\ -4.5 & 13.5 & -13.5 & 4.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

Matrix Form: y(t)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 1/9 & 1/27 \\ 1 & 2/3 & 4/9 & 8/27 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_{y,0} \\ a_{y,1} \\ a_{y,2} \\ a_{y,3} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Solution: y(t)

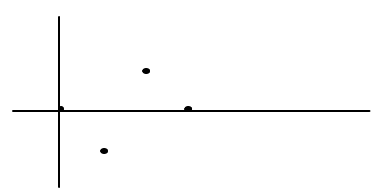
$$\begin{bmatrix} a_{y,0} \\ a_{y,1} \\ a_{y,2} \\ a_{y,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5.5 & 9 & -4.5 & 1 \\ 9 & -22.5 & 18 & -4.5 \\ -4.5 & 13.5 & -13.5 & 4.5 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -13.5 \\ 40.5 \\ -27 \end{bmatrix}$$

Example

- The interpolating polynomial that satisfies:
  p(0)=(0,0), p(1/3)=(1,-1),
  p(2/3)=(2,1), and p(1)=(3,0)
- Is:
  $x(t) = 3t$
  $y(t) = -13.5t + 40.5\,t^2 - 27\,t^3$

Example

- $x(t) = 3t$, $y(t) = -13.5t + 40.5\,t^2 - 27\,t^3$



Problems with interpolating polynomial

- Not smooth at joins

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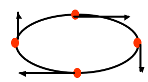
Hermite Curves

- Specify endpoints of curve: p(0) and p(1)
- Specify tangent of curve at the endpoints: p′(0) and p′(1)



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Cubic Curve: Hermite

- The points (x(t),y(t)), $t \in [0,1]$, where

  $x(t)=a_{x,0} + a_{x,1}t + a_{x,2}t^2 + a_{x,3}t^3$

  $y(t)=a_{y,0} + a_{y,1}t + a_{y,2}t^2 + a_{y,3}t^3$

- The user specifies the endpoints (x(0),y(0)) and (x(1),y(1)) + tangent vectors at the endpoints (x′(0),y′(0)) and (x′(1),y′(1))

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Example

- p(0)=(0,0), p(1)=(3,0), p′(0)=(1,1), p′(1)=(-1,1)

- We want to find the coefficients of :

  $x(t) = a_{x,0} + a_{x,1}t + a_{x,2}t^2 + a_{x,3}t^3$

  $(x'(t) = a_{x,1} + 2a_{x,2}t + 3a_{x,3}t^2)$

  given:

  $x(0)= \quad 0= a_{x,0}$

  $x(1)= \quad 3= a_{x,0} + a_{x,1} + a_{x,2} + a_{x,3}$

  $x'(0)= \quad 1= a_{x,1}$

  $x'(1) = -1= a_{x,1} + 2a_{x,2} + 3a_{x,3}$

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Matrix Form: x(t)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_{x,0} \\ a_{x,1} \\ a_{x,2} \\ a_{x,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}$$

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Solution: x(t)

$$\begin{bmatrix} a_{x,0} \\ a_{x,1} \\ a_{x,2} \\ a_{x,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & 1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 6 \\ -6 \end{bmatrix}$$

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Matrix Form: y(t)

1	0	0	0	$a_{y,0}$		0
1	1	1	1	$a_{y,1}$	=	0
0	1	0	0	$a_{y,2}$		1
0	1	2	3	$a_{y,3}$		1

Solution: y(t)

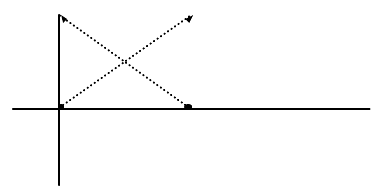
$a_{y,0}$		1	0	0	0	0		0
$a_{y,1}$	=	0	0	1	0	0		1
$a_{y,2}$		-3	3	-2	1	1	=	-1
$a_{y,3}$		2	-2	1	1	1		2

Example: Hermite

- The Hermite polynomial that satisfies:
  $p(0)=(0,0)$, $p(1)=(3,0)$,
  $p'(0)=(1,1)$, and $p'(1)=(-1,1)$
- Is:
  $x(t) = t + 6t^2 - 6t^3$
  $y(t) = t - t^2 + 2t^3$

Example: Hermite

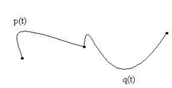
- $x(t) = t + 6t^2 - 6t^3$, $y(t) = t - t^2 + 2t^3$



Problems with Hermite polynomial

- User has to specify a derivative.

Parametric Continuity



- C^0 continuity: $p(1)=q(0)$
- C^1 continuity: $p(1)=q(0)$ & $p'(1)=q'(0)$

⋮

- C^n continuity: $p^{(i)}(1)=q^{(i)}(0)$ for $i \leq n$

Geometric Continuity

- G^0 continuity: p(1)=q(0)
- G^1 continuity: p(1)=c(0) & $p'(1)=cq'(0)$

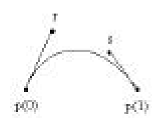
Continuity: $C^i \Rightarrow G^i$

- Interpolating Polynomial: C^0
- Hermite Polynomial: C^1
- Bezier Polynomial: C^0 and G^1

Cubic Curve: Bezier

- The points (x(t),y(t)), $t \in [0,1]$, where
  $x(t)=a_{x,0} + a_{x,1}t + a_{x,2}t^2 + a_{x,3}t^3$
  $y(t)=a_{y,0} + a_{y,1}t + a_{y,2}t^2 + a_{y,3}t^3$
- The user specifies the endpoints (x(0),y(0)) and (x(1),y(1)) + two control points u,v

Bezier



- Approximations: $p'(0)=3(r-p(0))$ and $p'(1)=3(p(1)-s)$

Example: Bezier

- p(0)=(0,0), p(1)=(3,0), r=(1,1), s=(2,1)
- We want to find the coefficients of :
  $x(t) = a_{x,0} + a_{x,1}t + a_{x,2}t^2 + a_{x,3}t^3$
  $(x'(t) = a_{x,1} + 2a_{x,2}t + 3a_{x,3}t^2\ )$

given:

$x(0)=\quad 0= a_{x,0}$
$x(1)=\quad 3= a_{x,0} + a_{x,1} + a_{x,2} + a_{x,3}$
$x'(0)=\quad 3(1-0)=3= a_{x,1}$
$x'(1) =\quad 3(3-2)=3= a_{x,1} + 2a_{x,2} + 3a_{x,3}$

Matrix Form: x(t)

1	0	0	0	$a_{x,0}$		0
1	1	1	1	$a_{x,1}$	=	3
0	1	0	0	$a_{x,2}$		3
0	1	2	3	$a_{x,3}$		3

Solution: x(t)

$$\begin{bmatrix} a_{x,0} \\ a_{x,1} \\ a_{x,2} \\ a_{x,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & 1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \\ 0 \end{bmatrix}$$

Matrix Form: y(t)

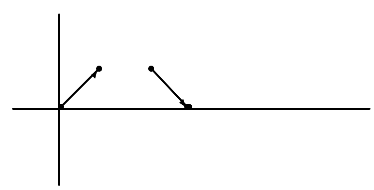
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_{y,0} \\ a_{y,1} \\ a_{y,2} \\ a_{y,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Solution: y(t)

$$\begin{bmatrix} a_{y,0} \\ a_{y,1} \\ a_{y,2} \\ a_{y,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & 1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}$$

Example: Bezier

- $x(t) = 3t + 6t^2$, $y(t) = t - 3t^2$



Property of Bezier

- The Bezier curve lies within the convex hull of the control points p(0),p(1),r,s.

Assignment 6

- User specifies p(0),p′(0),p″(0), and p(1)
- What equations have to be satisfied?
- What is the matrix form?
- What is the solution?

B-spline

- Read the text

Bezier Curves in OpenGL

- glMap1{fd}(args) see WNDS pp 500-501

Next Time

- Modeling curved surfaces