CS140: Algorithms

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Lecture 1
1/18/01

Last class

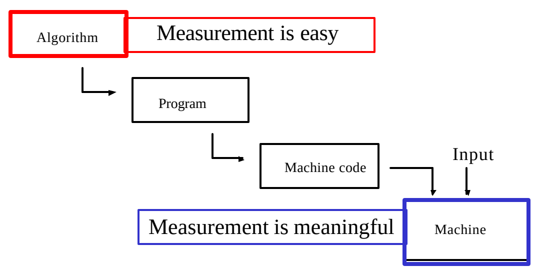
The two important questions we consider in CS140:

- Is the computational procedure correct?
- **Is the algorithm fast?**

How do we measure speed?

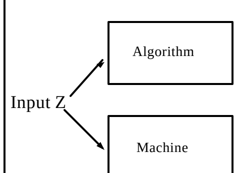
- **What to measure**
- Big-O notation/rate of growth
- Loop counting
- Series

Running Time
Where to measure?



A useful assumption

T_A and T_M differ by no more than a multiplicative constant



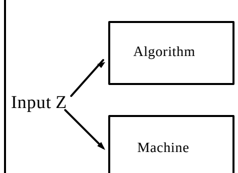
$T_A(Z)$: number of steps of algorithm on input Z

$T_M(Z)$: number of steps of machine on input Z

More formally

There is some constant c such that for any input Z

$T_A(Z)/c \le T_M(Z) \le cT_A(Z)$



$T_A(Z)$: number of steps of algorithm on input Z

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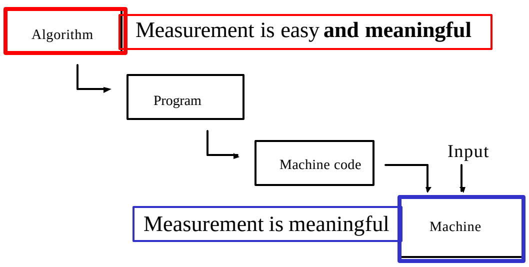
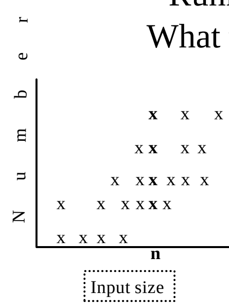
Running Time
Where to measure?

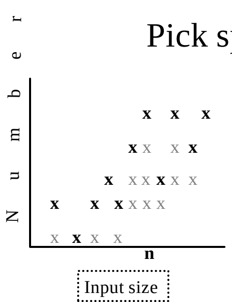


Running Time:
What to measure?



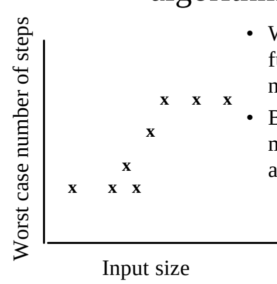
- Run time depends on input size
- Run time can vary on different inputs of size n.

Pick special case



- Run time depends on input size
- Run time can vary on different inputs of size n.
- Choose case:
  - Worst case (show in bold)
  - Best case
  - Average case
  - Etc.

Worst case performance of algorithm A



- We can compute this function at a finite number of points .
- Better yet, we can model this function for all input sizes.

A general problem …

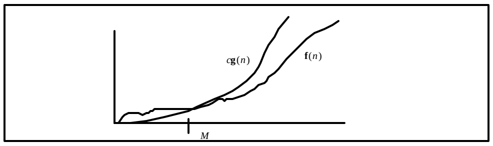
- Question: How can we give a succinct description of an arbitrary function?
- Answer: Big-O notation.

Today

- What to measure
- **Big-O notation/rate of growth**
- Loop counting
- Series

Upper Bounds

- $\mathbf{f}$:N→N and $\mathbf{g}$:N→N are positive-valued, monotonically increasing functions.
- O($\mathbf{g}(n)$) = {$\mathbf{f}(n)$: there are constants c and M such that $\mathbf{f}(n) \leq c\,\mathbf{g}(n)$ for all $n \geq M$}



Proving f(n)=O(g(n))

Consider h(n)=f(n)/g(n) as n goes to infinity

- h(n) converges
- h(n) diverges
- h(n) oscillates

Some useful observations about Big-O

- If f(n)/g(n) converges then f(n) ___O(g(n)).
- If f(n)/g(n) diverges then f(n)___O(g(n)).
- If f(n)/g(n) oscillates then f(n)____O(g(n)).

Logarithms

For which pairs f(n), g(n) is f(n)=O(g(n))?

lg n
$\log n^2$
$\lg^2 n$
lg 10000n

Example

$\lim_{n\to\infty} \lg n/\log n^2 = (\lg 10)/2$

(Useful observation: $\log n^2 = (2/\lg 10) \lg n$)

Limits

1. $\lim_{n\to\infty} \log n^2 / \lg n$
2. $\lim_{n\to\infty} \lg n / \lg^2 n$
3. $\lim_{n\to\infty} \lg^2 n / \lg n$
4. $\lim_{n\to\infty} \lg n / \lg 10000 n$

Polynomials

For which pairs f(n), g(n) is f(n)=O(g(n))?

n

n^2

$1000n^2 + n$

Exponentials

For which pairs f(n),g(n) is f(n)=O(g(n))?

2^n

3^n

$2^{(n^2)}$

$(2^n)^2$

Some rules of thumb

- Polylogs are slower growing than polynomials

  For any k,j > 0:

  $\log^j n = O(n^k)$ and $n^k \neq O(\log^j n)$
- Polynomials are slower growing than exponentials

  For any k>0 and r>1:

  $n^k = O(r^n)$ and $r^n \neq O(n^k)$

L'hospital's rule

- $\lim_{\to\infty} \log^j n \ / \ n^k = 0$
- $n^k \ / \log^j n$ diverges as n goes to infinity

Polynomially bounded functions

f(n) is polynomially bounded if there is a constant k such that $f(n) = O(n^k)$

Logs, Polys, and Exps

Which of the following functions are polynomially bounded?

$\log n$

n^3

2^n

Other functions

- Factorial: $n! = n\,(n-1)!,\ 0!=1$
- Tower of 2s: $2\uparrow\uparrow n = 2^{2\uparrow\uparrow(n-1)},\ 2\uparrow\uparrow 0 = 1$
- Iterated log: $\text{Log}^*(n) = m$ such that $2\uparrow\uparrow(m-1) < n \le 2\uparrow\uparrow n$
- Ceil-ceil: $\lceil\lceil n \rceil\rceil = 2^m$ such that $m-1 < \lg n \le m$

Logs, polys, exps, and others

Compare the rates of growth of the following functions:

$\lg n,\ \ n^3,\ \ 2^n,\ \ n!,\ \ 2\uparrow\uparrow n,\ \ \log^*(n),\ \lceil\lceil n \rceil\rceil$

Another useful observation

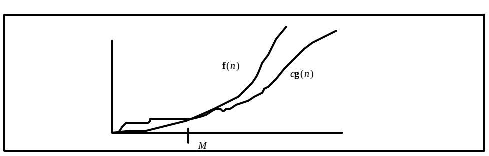
- If $f(n)/g(n)$ diverges then so does $2^{f(n)}/2^{g(n)}$
- If $\lg(f(n))/\lg(g(n))$ diverges then so does $f(n)/g(n)$

Beyond O

real numbers	functions
• $\le$	• O
• $\ge$	• Ω
• $=$	• Θ
• $<$	• o
• $>$	• ω

Lower Bounds

- $\mathbf{f}:N \to N$ and $\mathbf{g}:N \to N$ are positive-valued, monotonically increasing functions.
- $\Omega(\mathbf{g}(n)) = \{\mathbf{f}(n)$: there are constants c and M such that $\mathbf{f}(n) \ge c\,\mathbf{g}(n)$ for all $n \ge M\}$



Definition: Θ

$f(n) = \Theta(g(n))$ if the following hold:

1. $f(n) = O(g(n))$, and
2. $f(n) = \Omega(g(n))$

Definition: little-o, little-ω

- $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$
- $f(n) = \omega(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = \infty$

Logs, polys, exps, and others

Compare the following functions. Which of O, Ω, Θ, o, and ω apply?

$\lg n$, n^3, 2^n, $n!$, $2\uparrow\uparrow n$, $\log^*(n)$, $\lceil\lceil n\rceil\rceil$

A slight twist…

Is $f(2n) = O(f(n))$?

1. $f(n) = 1$: Is $2n=O(n)$?
2. $f(n) = 3n$: Is $6n=O(3n)$?
3. $f(n) = n^2$: Is $4n^2=O(n^2)$?
4. $f(n) = 2^n$: Is $4^n=O(2^n)$?
5. $f(n) = n!$: Is $(2n)!=O(n1)$?

Today

- **What to measure**
- **Big-O notation/rate of growth**
- Loop counting
- Series

CS140 pragmatism

What is the asymptotic behavior of the worst-case running time of the algorithm?

CS140 pragmatism

Big-O

What is the **asymptotic behavior** of the worst-case running time of the algorithm?

CS140 pragmatism

What is the asymptotic behavior of the **worst-case** running time of the algorithm?

Special case input

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CS140 pragmatism

What is the asymptotic behavior of the worst-case **running time** of the algorithm?

Chosen resource

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CS140 pragmatism

What is the asymptotic behavior of the worst-case running time of the **algorithm**?

Remember our assumption

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Run time bounds for algorithm $\mathbf{A}$

The running time of $\mathbf{A}$ is $O(n^3)$.

$\Updownarrow$

The worst case running time of $\mathbf{A}$ is $O(n^3)$.

$\Updownarrow$

$\mathbf{A}$ is $O(n^3)$.

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Rate of growth of common functions

- Review of properties/notation
- See CLR pp 32-37 for details

KNOW THIS STUFF

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Today

- How should we measure the speed of an algorithm?
- Big-O notation/rate of growth
- **Loop counting**
- Series

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Types of Algorithms

- Recursive Algorithm: one that calls itself
- Purely Iterative Algorithm: one that doesn't

Run Time Analysis

- **Iterative algorithm** ® **Loop counting**
- Recursive algorithm → Recurrence relations

Iterative Sorting Algorithms

- Insertion-sort
- Bubble-sort
- Modified Bubble-sort

Insertion-sort(S)

(in pseudo-code)

S is an array of n integers: S(1), S(2), …, S(n)

```
For j = 2 to n
  key = S(j)
  i = j-1
  While i > 0 and S(i) > key
    S(i+1) = S(i--)
  S(i+1) = key
```

C decrement operator

Correctness

- Inductive proof with loop invariant:

  When the **for loop** executes for the k^{th} time, S(1), S(2), …, S(k) are sorted in ascending order.

Loop Counting: Insertion-sort(S)

$\Sigma_{j=2..n}$ (1 → For j = 2 to n

+1 → key = S(j)

+1 → i = j-1

+ $\Sigma_{i=0..j-1}$ (1 → While i>0 & S(i)>key

+1) → S(i+1) = S(i--)

+1) → S(i+1) = key

$$\boxed{\Sigma_{j=2..n}(4+\Sigma_{i=0..j-1}2) = O(n^2)}$$

Bubble-sort(S)

```
Bubble-sort(S)
    For i=n down to 2
        For j=1 to i-1
            If S(j) >S(j+1) then swap(S(j), S(j+1) )
    Return
```


Correctness

- Inductive Proof with loop invariant:

  When the i-loop completes its k^{th} execution,
  - $S(n-k+1), S(n-k+2), \ldots, S(n)$ is sorted in ascending order, and
  - the $\max(S(1), \ldots, S(n-k)) \leq S(n-k+1)$.

Does Bubble-sort do too much work?

1, 3, 2, 4, 5
1, 2, 3, 4, 5
1, 2, 3, 4, 5
1, 2, 3, 4, 5
1, 2, 3, 4, 4

- Repeat on smaller list

Modified Bubble-sort

```
Modified-Bubble-sort(S)
    SWAP=T
    For i=n down to 2
        If SWAP=F then return
        SWAP=F
        For j=1 to i-1
            If S(j)>S(j+1) then swap(S(j),S(j+1)) and set
              SWAP=T
    Return
```


Example

1, 3, 2, 4, 5
1, 2, 3, 4, 5
1, 2, 3, 4, 5

- Repeat on smaller list unless no swaps are made

Loop counting: M-Bubble-sort

Modified-Bubble-sort(S)

$1 +$ → SWAP=T

$\Sigma_{i=2..n}($ 1+ → For i=n down to 2

1+ → If SWAP=F then return

1+ → Flag=F

$\Sigma_{j=1..i-1}($ 1+ → For j=1 to i-1

3) → If $s_j > s_{j+1}$ then swap(s_j, s_{j+1}) and set SWAP=T

Return

$$\boxed{1+\Sigma_{i=2\ldots n}(3+\Sigma_{j=1\ldots i-1} 4)) = O(n^2)}$$

Summation

$\Sigma_{i=2...n}\ \Sigma_{j=1...i-1}c \quad = c\Sigma_{i=2...n}(i-1)$

$= c(\Sigma_{i=1...n}(i-1)) - c$

$= c(\Sigma_{i=1...n} i - \Sigma_{i=1...n} 1) - c$

$= c(n(n+1)/2 - n) - c$

$=$ **$O(n^2)$**

Series

- A series is a summation of terms
- Common series
  - Arithmetic series: $1+2+...+n$
  - Geometric series: $1+a+a^2+...+a^n$

Series
Things we want to do:

- **Solve exactly**
- Bound above or below
- Prove that a solution (or bound) is correct

Closed form solutions to some common series

- $f(n) = 1+2+ ... + n = n(n+1)/2$
- $f(n) = 1^2 + 2^2 + \dots + n^2 = (2n^3 + 3n^2 + n)/6$
- $f(n) = 1+a+a^2+...+a^n = n$ if $a=1$
  $= (a^{n+1}-1)/(a-1)$ else
- $f(n) = 1+a+a^2+\dots = 1/(1-a)$ if $0 \le a<1$

Series
Things we want to do:

- Solve exactly
- **Bound above or below**
- Prove that a solution (or bound) is correct

Upper Bounds on series

For any constant k:

$$\Sigma_{i=1..n} i^k \le \Sigma_{i=1..n} n^k$$
$$= (n^{k+1})$$
$$= O(n^{k+1})$$

Is this a good upper bound?

Lower Bounds on series

For any constant k:

$$\Sigma_{i=1..n} i^k \geq \Sigma_{i=\lfloor n/2 \rfloor \ldots n} i^k$$
$$\geq \Sigma_{i=\lfloor n/2 \rfloor \ldots n} \lfloor n/2 \rfloor^k$$
$$\geq \lfloor n/2 \rfloor^{k+1}$$
$$= \Omega(n^{k+1})$$

So $\Sigma_{i=1..n} i^k = \Theta(n^{k+1})$

Series
Things we want to do:

- Solve exactly
- Bound above or below
- **Prove that a solution (or bound) is correct**

Proving correctness

- Claim: $\Sigma_{i=1..n} i^2 = (2n^3 + 3n^2 + n)/6$
- Claim holds for n=1.
- If the claim holds for n then it holds for n+1:

$$\Sigma_{i=1..n+1} i^2 = (n+1)^2 + \Sigma_{i=1..n} i^2$$
$$= (n+1)^2 + (2n^3 + 3n^2 + n)/6$$
$$= (2(n+1)^3 + 3(n+1)^2 + (n+1))/6$$

Next time

- Recursive algorithms
- Recurrence relations