

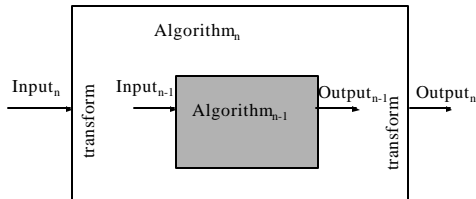
CS140: Algorithms

Z Sweedyk
Lecture 8
3/1/01

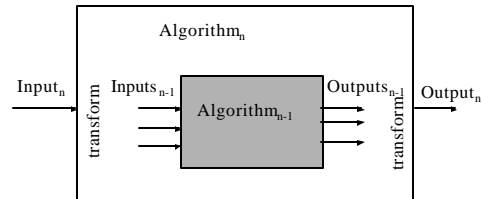
Algorithm Design Techniques

- Induction
- Reduction
- **Divide and Conquer**

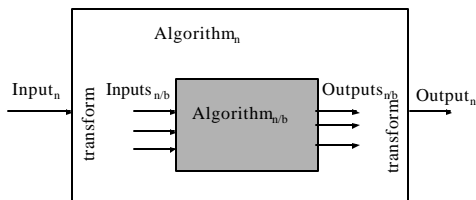
Induction: Algorithm_n



Induction: Algorithm_n



Divide and Conquer: Algorithm_n



Sorting

- Insertion Sort: $T(n) = T(n-1) + cn = O(n^2)$
 - Sort first $n-1$ elements
 - Insert element n into sorted list
- Merge Sort: $T(n) = 2T(n/b) + cn = O(n \lg(n))$
 - Sort first $n/2$ elements
 - Sort last $n/2$ elements
 - Merge sorted lists into one sorted list

Matrix Multiplication

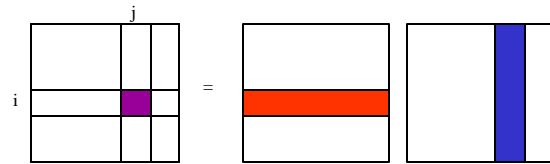
- Input: Two $n \times n$ matrices A, B
- Output: The product AB

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Reminder



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Example

- Input:

-1	3
2	4

2	-3
2	1
- Output:

4	6
12	-2

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Algorithm

```

For i = 1 to n
  For j = 1 to n
    C[i,j]=0
    For k = 1 to n
      C[i,j] = C[i,j] + A[i,k]B[k,j]
    
```

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Operation Count

```

For i = 1 to n
  For j = 1 to n
    C[i,j]=0
    For k = 1 to n
      C[i,j] = C[i,j] + A[i,k]B[k,j]
    
```

Requires n^3 multiplications and $n^2(n-1)$ additions.

(Note the input size is $2n^2$.)

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Divide and Conquer: $n=2^m$

- Input:

A_{11}	A_{12}
A_{21}	A_{22}

B_{11}	B_{12}
B_{21}	B_{22}
- Output:

C_{11}	C_{12}
C_{21}	C_{22}

 $C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j}$

$$T(2n^2) = 8T(2(n/2)^2) + cn^2 = O(n^3)$$

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Can we do better?

- $T(2n^2) = aT(2(n/2)^2) + cn^2 = o(n^3)$ if $a < 8$
- Strassen's algorithm:
 $a=7$ and $T(n) = O(n^{\log_2 7})$

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Strassen's Algorithm

Base Case: 2x2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} ae+bf & ag+bh \\ ce+df & cg+dh \end{bmatrix}$$

Is there hope?

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Rephrase

Call this matrix A Call this vector v

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} ae+bf \\ ce+df \\ ag+bh \\ cg+dh \end{bmatrix}$$

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A little magic ...

We can compute Av with 7 scalar multiplications!

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B and C

$$\begin{matrix} B & C & B+C \\ \begin{bmatrix} b & b & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & c \\ 0 & 0 & c & c \end{bmatrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{matrix}$$

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B and C

$$\begin{matrix} B & C & B+C \\ \begin{bmatrix} b & b & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & c \\ 0 & 0 & c & c \end{bmatrix} & \begin{bmatrix} b & b & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & c & c \\ 0 & 0 & c & c \end{bmatrix} \end{matrix}$$

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B, C and D

b	b	0	0
b	b	0	0
0	0	c	c
0	0	c	c

0	0	0	0
c-b	0	0	c-b
b-c	0	0	b-c
0	0	0	0

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B, C and D

b	b	0	0
b	b	0	0
0	0	c	c
0	0	c	c

0	0	0	0
c-b	0	0	c-b
b-c	0	0	b-c
0	0	0	0

b	b	0	0
c	b	0	c-b
b-c	0	c	b
0	0	c	c

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B, C, D and E

b	b	0	0
c	b	0	c-b
b-c	0	c	b
0	0	c	c

a-b	0	0	0
0	0	0	0
c-b	0	a-c	0
0	0	0	0

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B, C, D and E

b	b	0	0
c	b	0	c-b
b-c	0	c	b
0	0	c	c

a-b	0	0	0
0	0	0	0
c-b	0	a-c	0
0	0	0	0

a	b	0	0
c	b	0	c-b
0	0	a	b
0	0	c	c

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B, C, D, E and F

a	b	0	0
c	b	0	c-b
0	0	a	b
0	0	c	c

0	0	0	0
0	d-b	0	b-c
0	0	0	0
0	0	0	d-c

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B, C, D, E and F

a	b	0	0
c	b	0	c-b
0	0	a	b
0	0	c	c

0	0	0	0
0	d-b	0	b-c
0	0	0	0
0	0	0	d-c

a	b	0	0
c	d	0	0
0	0	a	b
0	0	c	d

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$$A = B + C + D + E + F$$

a	b	0	0
c	d	0	0
0	0	a	b
0	0	c	d

e
f
g
h

$$=$$

ae+bf
ce+df
ag+bh
cg+dh

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A little magic...

Claim: We can compute the products Bv , Cv , Dv , Ev , and Fv with 7 scalar multiplications

So we can compute Av with 7 scalar multiplications

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Bv : 1 multiplication

b	b	0	0
b	b	0	0
0	0	0	0
0	0	0	0

e
f
g
h

$$=$$

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Bv : 1 multiplication

b	b	0	0
b	b	0	0
0	0	0	0
0	0	0	0

e
f
g
h

$$=$$

b(e+f)
b(e+f)
0
0

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Cv : 1 multiplication

0	0	0	0
0	0	0	0
0	0	c	c
0	0	c	c

e
f
g
h

$$=$$

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Cv : 1 multiplication

0	0	0	0
0	0	0	0
0	0	c	c
0	0	c	c

e
f
g
h

$$=$$

0
0
c(g+h)
c(g+h)

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Dv: 1 multiplication

$$\begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline c-b & 0 & 0 & c-b \\ \hline b-c & 0 & 0 & b-c \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}
 \begin{array}{c} v \\ e \\ f \\ g \\ h \end{array}
 =
 \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

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Dv: 1 multiplication

$$\begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline c-b & 0 & 0 & c-b \\ \hline b-c & 0 & 0 & b-c \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}
 \begin{array}{c} v \\ e \\ f \\ g \\ h \end{array}
 =
 \begin{array}{|c|} \hline 0 \\ \hline (c-b)(e+h) \\ \hline (b-c)(e+h) \\ \hline 0 \\ \hline \end{array}$$

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Ev: 2 multiplication

$$\begin{array}{|c|c|c|c|} \hline a-b & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline c-b & 0 & a-c & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}
 \begin{array}{c} v \\ e \\ f \\ g \\ h \end{array}
 =
 \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

Note: $(c-b)e+(a-c)g=(a-c)(g-e)+(a-b)e$

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Ev: 2 multiplication

$$\begin{array}{|c|c|c|c|} \hline a-b & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline c-b & 0 & a-c & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}
 \begin{array}{c} v \\ e \\ f \\ g \\ h \end{array}
 =
 \begin{array}{|c|} \hline (a-b)e \\ \hline 0 \\ \hline (c-b)e+(a-c)g \\ \hline 0 \\ \hline \end{array}$$

Note: $(c-b)e+(a-c)g=(a-c)(g-e)+(a-b)e$

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Fv: 2 multiplication

$$\begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & d-b & 0 & b-c \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & d-c \\ \hline \end{array}
 \begin{array}{c} v \\ e \\ f \\ g \\ h \end{array}
 =
 \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

Note: $(d-b)(f-h)=(d-b)f+(b-c)h-(d-c)h$

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Fv: 2 multiplication

$$\begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & d-b & 0 & b-c \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & d-c \\ \hline \end{array}
 \begin{array}{c} v \\ e \\ f \\ g \\ h \end{array}
 =
 \begin{array}{|c|} \hline 0 \\ \hline (d-b)f+(b-c)h \\ \hline 0 \\ \hline (d-c)h \\ \hline \end{array}$$

Note: $(d-b)(f-h)=(d-b)f+(b-c)h-(d-c)h$

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Strassen's Divide and Conquer

- Input:

A_{11}	A_{12}
A_{21}	A_{22}

B_{11}	B_{12}
B_{21}	B_{22}
- Output:

C_{11}	C_{12}
C_{21}	C_{22}

 $C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j}$

$$T(2n^2) = 7T(2(n/2)^2) + cn^2 = O(n^{2.81})$$

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Closest Pair

- Input: Set of points $n \geq 2$ on the plane $\{(x_i, y_i) \mid 1 \leq i \leq n\}$
- Output: Pair of closest points (under Euclidian distance)

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Closest Pair Example

- Input: $\{(1,1), (2,2), (10,1), (10,2)\}$
- Output: (10,1) and (10,2)

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Closest Pair – Algorithm Naïve Approach

- Compute pair-wise distances
- Choose smallest

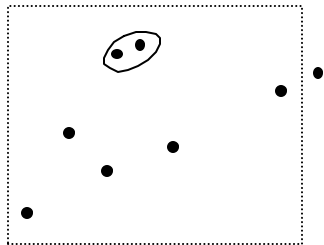
Requires $O(n^2)$ time

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Simple Inductive Approach Find Closest Pair in $\{p_1, p_2, \dots, p_{n-1}\}$

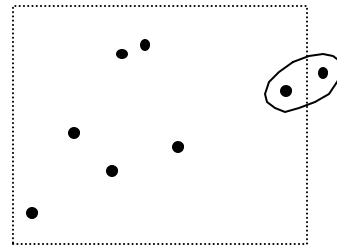


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Simple Inductive Approach cont. Find Closest point to p_n

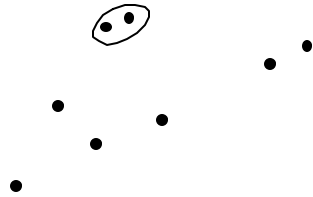


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Simple Inductive Approach cont.
Choose best of best



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Simple Inductive Approach

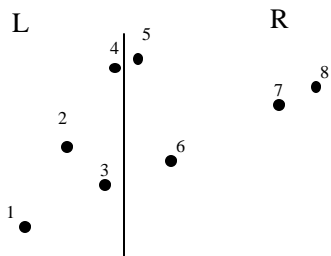
- $T(n) = T(n-1) + O(n) = O(n^2)$

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Divide and Conquer
Partition by x-coordinate

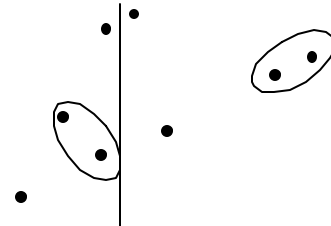


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Divide and Conquer cont.
Solve sub-problems

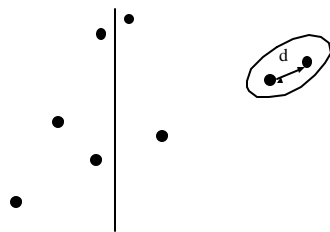


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Divide and Conquer cont.
Choose best of best

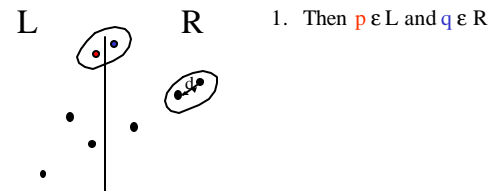


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What have we missed?
Suppose p and q are closer than d.

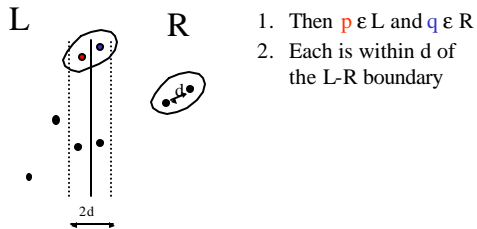


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What have we missed?
Suppose p and q are closer than d .



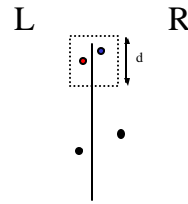
1. Then $p \in L$ and $q \in R$
2. Each is within d of the L-R boundary

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What have we missed?
Suppose p and q are closer than d .



1. Then $p \in L$ and $q \in R$
2. They are within d of the L-R boundary
3. Their y-coordinates differ by less than d

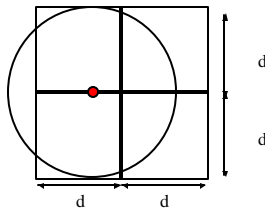
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Objective: $n \lg n$ vs. n^2

- Sort ($n \lg n$) then search (constant)

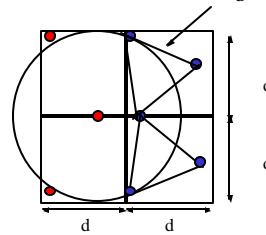


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If $d(p,q) < d$ then
 $|\text{rank}(p_y) - \text{rank}(q_y)| < 4$



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Closest-pair(P)

- Let M be the median p_x for $p \in P$
- $P_1 = \{p \in P \mid p_x \leq M\}$, $P_2 = \{p \in P \mid p_x > M\}$
- Let (r,s) be the closest of the pairs
Closest-pair(P_1) and Closest-pair(P_2)
- Let $Q = \{p \in P \mid d(r,s) > |p_x - M|\}$
- Sort Q by y-coordinate.
- Find closest pair p_i and $p_j \in Q$, where $0 < |i-j| < 4$.
- Return best pair found

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Running time of Closest Pair

- $T(n) = 2T(n/2) + n \lg(n) = O(n \lg(n))$

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