

CS140: Algorithms

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Lecture 16
4/10/01

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1

NP-completeness

Problem A is NP-Complete if

- A is in NP
- A is NP-hard

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2

A \in NP

A is a decision problem and its yes-instances can be described as

$$\{x \mid \exists y \text{ such that } P(x,y)\}$$

for a polynomial-time checkable predicate P.

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3

Clique

- Input: A graph G and an integer K
- Question: Does G contain a fully-connected subgraph with K or more vertices?

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4

Clique is in NP

Proof: Show how to express the yes-instances of Clique as $\{x \mid \exists y \text{ such that } P(x,y)\}$.

- x: G,K
- y: A subset W of the vertices of G
- $P(x,y)=P(G,W)$:
 - $\|W\| \geq K$
 - Each pair of vertices of W is connected by an edge in G

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5

Independent Set

- Input: A graph G and an integer K
- Question: Does G have K or more vertices that mutually non-adjacent?

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6

Independent is in NP

Proof: Show how to express the yes-instances of Independent Set as $\{x \mid \exists y \text{ such that } P(x,y)\}$.

- x : G, K
- y : A subset W of the vertices of G
- $P(x,y)=P(G,W)$:
 - $\|W\| \geq K$
 - No pair of vertices in W is connected by an edge in G

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7

Vertex Cover

- Input: A graph G and an integer K
- Question: Does G have K or fewer vertices that touch every edge of G ?

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8

Vertex Cover is in NP

Proof: Show how to express the yes-instances of Vertex Cover as $\{x \mid \exists y \text{ such that } P(x,y)\}$.

- x : G, K
- y : A subset W of the vertices of G
- $P(x,y)=P(G,W)$:
 - $\|W\| \geq K$
 - Every edge of G is incident to some vertex in W

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9

NP-completeness

Problem A is NP-Complete if

- A is in NP
- A is NP-hard

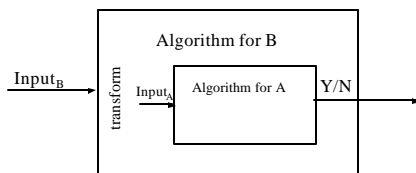
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10

A is NP-hard

For any problem $B \in \text{NP}$, $B \leq_p A$.



Note: If A can be solved in polynomial time, then so can every other problem in NP.

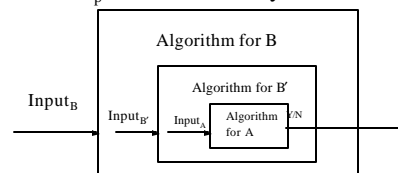
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11

To prove A is NP-hard

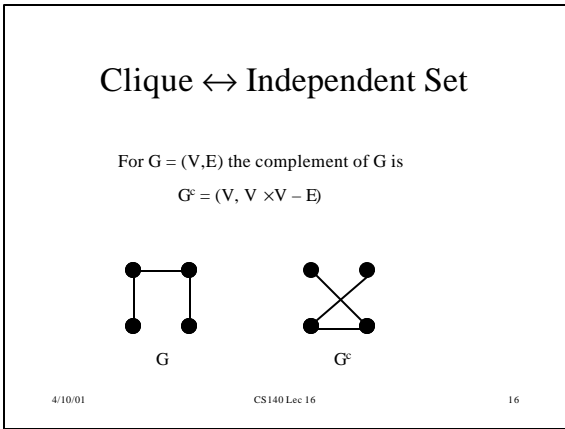
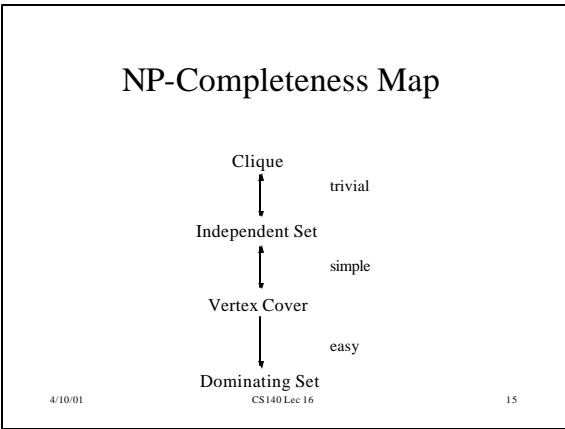
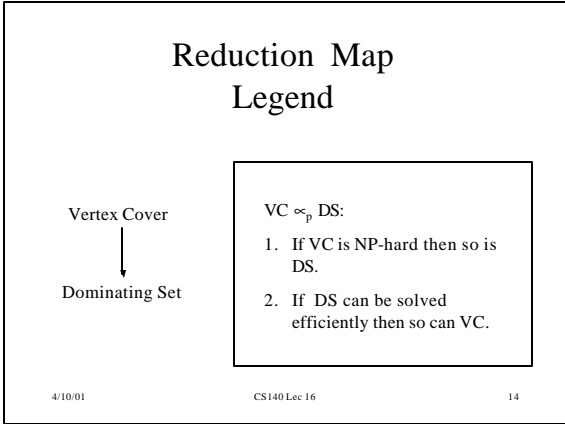
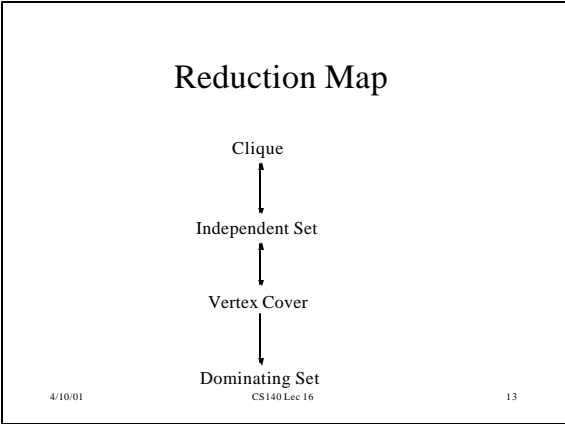
Find some NP-hard problem B' and show that $B' \leq_p A$. Then for any $B \in \text{NP}$, $B \leq_p A$.



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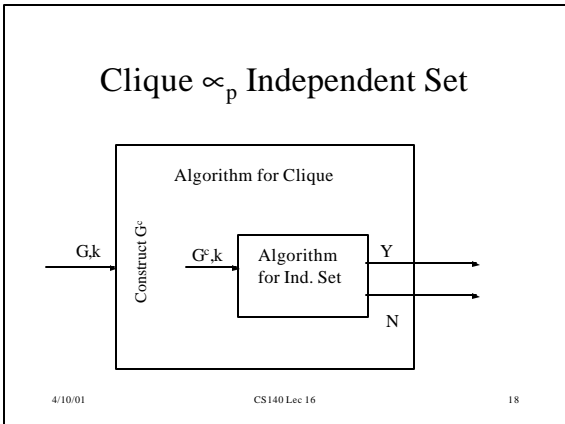
12



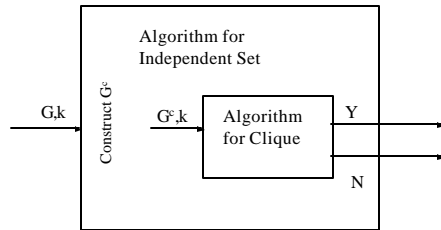
Clique \leftrightarrow Independent Set

- A clique in G is an independent set in G^c .
- A clique in G^c is an independent set in G .

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Independent Set \propto_p Clique



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19

$T_{\text{clique}}(n,m)$

$$T_{\text{clique}}(n,m) = n^2 + T_{\text{ind-set}}(n,n^2-m)$$

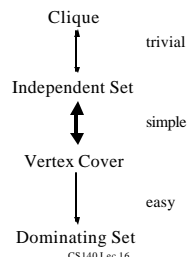
If $T_{\text{ind-set}}$ is polynomially-bounded then so is T_{clique} .

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20

NP-Completeness Map



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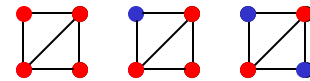
21

Vertex Cover \leftrightarrow Independent Set

For $G = (V,E)$:

$W \subseteq V$ is a **Vertex Cover** of G iff $V-W$ is an **Independent Set** of G .

Examples:



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22

Vertex Cover \leftrightarrow Independent Set

$W \subseteq V$ is a **Vertex Cover** of G iff $V-W$ is an **Independent Set** of G .

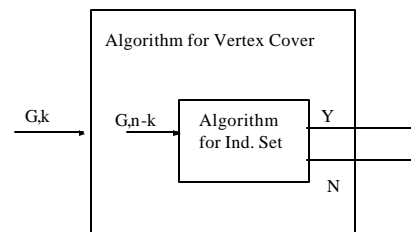
Every edge of G is incident to a vertex in W iff no edge of G has both endpoints in $V-W$.

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23

Vertex Cover \propto_p Independent Set



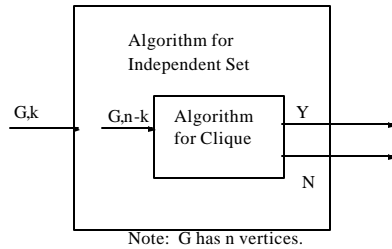
Note: G has n vertices.

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24

Independent Set \propto_p Vertex Cover

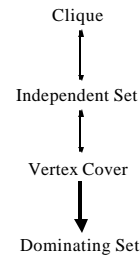


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25

Reduction Map

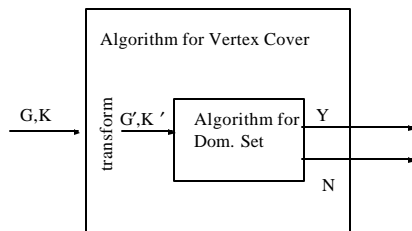


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26

Vertex Cover \propto_p Dominating Set

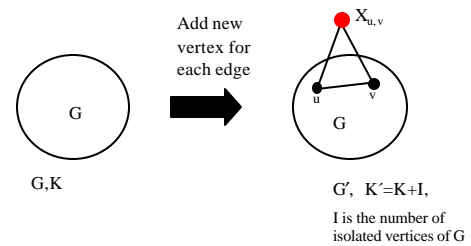


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27

Transform



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28

Claim

- G has a vertex cover of size K or less iff G' has a dominating set of size $K+I$ or less.

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29

Proof \Rightarrow

- Let W be a vertex cover of G containing K or fewer vertices. Let W' be the isolated vertices of G . Claim: $W \cup W'$ is a dominating set of G' containing $K+I$ or fewer vertices.
- Let v be a vertex of G' . We must show that either $v \in W \cup W'$ or v is adjacent to a vertex in $W \cup W'$.
 - If v is an isolated vertex then $v \in W'$ and we are done.
 - Suppose v is not an isolated vertex. Then we may assume that (u, v) is an edge of G . Since W is a vertex cover of G either $v \in W$ or $u \in W$. In either case we are done.

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Proof \Leftarrow

Let W be a dominating set of G' containing $K+I$ or fewer vertices.

Then there exists a dominating set of G' containing $K+I$ or fewer vertices all of which are vertices of G .

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31

Proof \Leftarrow

Let W be a dominating set of G' containing $K+I$ or fewer vertices. WLOG assume W only contains vertices of G .

Let W' be the isolated vertices of G . Then $W \cup W'$ is a vertex cover of G containing K or fewer vertices

Let $e=(u,v)$ be an edge of G . We must show that either u or v is in $W \cup W'$.

By our reductions, $X_{u,v}$ is a vertex of G' . By our previous argument, $X_{u,v}$ is not in W . Since W is a dominating set of G' , either u or v is in W . Neither u nor v is isolated so either u or v is in $W \cup W'$.

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32

Claim

- The reduction can be done in polynomial time.

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33

NP-completeness

Problem A is NP-Complete if

- A is in NP
- A is NP-hard

Show $B \leq_p A$, where B is NP-hard.

Where did it all begin?

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34

Cook's Theorem

- Cook's Theorem: SAT is NP-hard.
- SAT:
 - Input: A boolean expression E in conjunctive normal form.
 - Question: Does E have a satisfying truth assignment; i.e. a truth assignment under which E evaluates to true?

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35

Levin's Theorem

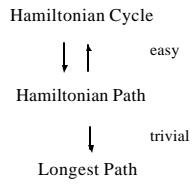
- Levin's Theorem: Tiling is NP-hard.

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36

Reduction Map II

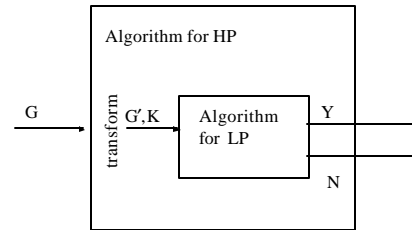


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37

$HP \propto_p LP$



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38

Transform

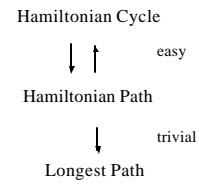
- By definition, G has a Hamiltonian Path iff it has a simple path of length $n-1$. ($n = \#$ vertices)
- So let $G' = G$ and $K = n-1$

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39

Reduction Map II

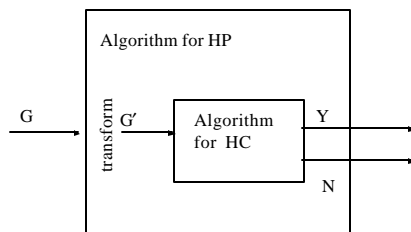


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40

$HC \propto_p HP$

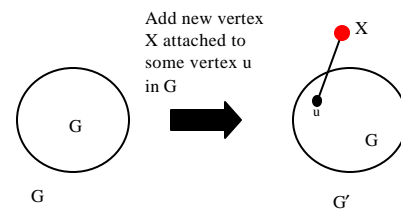


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41

Transform – Step 1

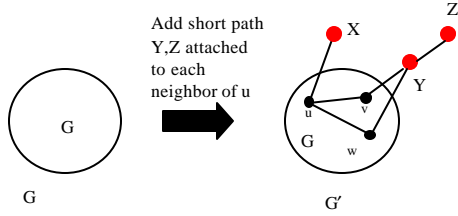


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42

Transform – Step 2



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43

Claim

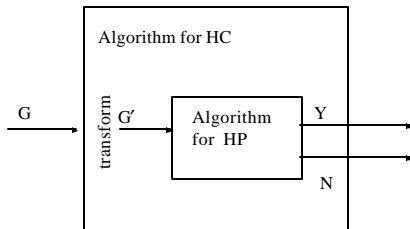
- G has a Hamiltonian Cycle iff G' has a Hamiltonian Path
- The reduction can be done in polynomial time

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Exercise: $HP \leq_p HC$

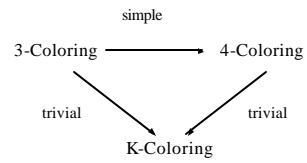


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45

Exercise: Reduction Map III



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46