

CS140: Algorithms

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Lecture 8

10/14/01

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Algorithm Design Techniques

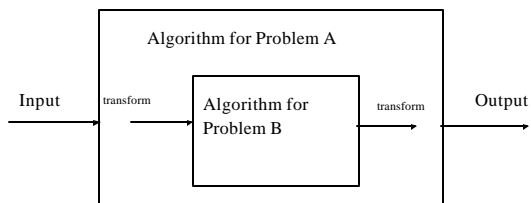
- Induction
- **Reduction**

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Reduction: $A \propto B$

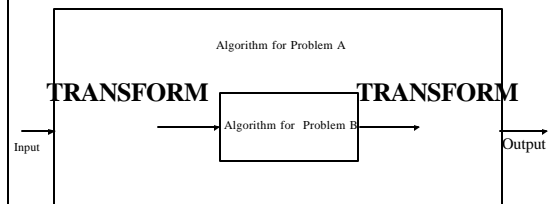


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Reduction: $A \propto B$



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Network Flow

Network Flow:

- Solving network flow problems
- Reductions to the network flow problem

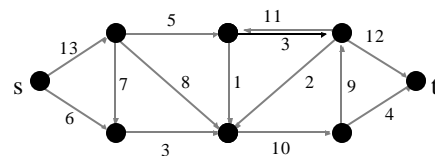
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Max Flow in a Network

- Input: Flow Network



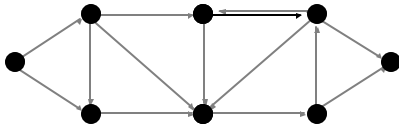
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Flow Network

- Directed graph



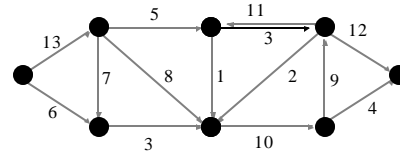
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Flow Network

- Directed graph with positive edge weights called **capacities**



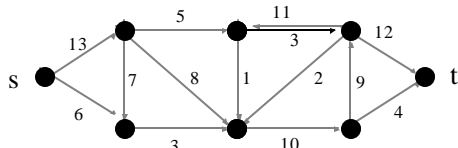
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Flow Network

- Directed graph with positive edge capacities and two special vertices s (the source) and t (the sink)



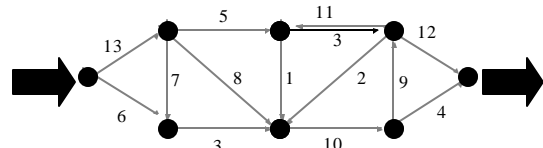
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Max Flow in a Network

- Output: Maximum flow that can be pumped through the network from s to t .



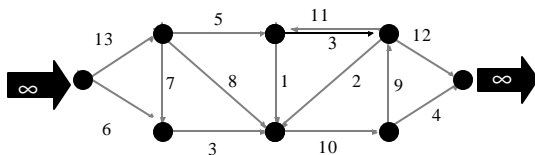
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The Rules

0. The source has infinite input capacity. The sink has infinite output capacity.



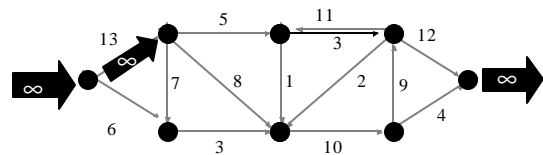
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The Rules

1. The total flow into a node must equal the total flow out of a node.



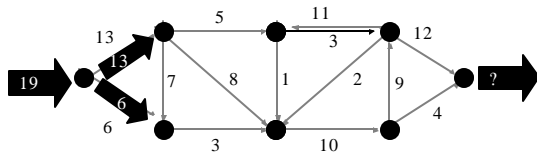
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The Rules

2. The flow along an edge cannot exceed its capacity. (No edge means 0 capacity.)



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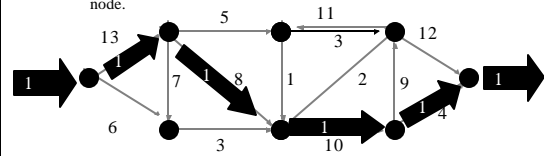
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What is a feasible flow?

Any flow that satisfies the rules.

1. The flow along an edge cannot exceed its capacity
2. The total flow into a node must equal the total flow out of a node.



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Conservation of Flow

- Flow in = Flow out (Rule 2)



- Flow into s = Flow out of t

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Network flow

- Flow into s = Flow out of t = Network flow

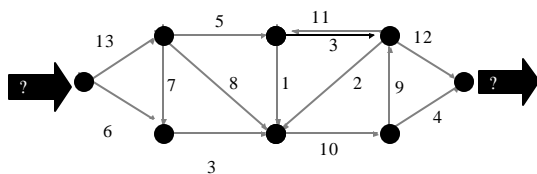
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What is the max flow?

1. The flow along an edge cannot exceed its capacity
2. The total flow into a node must equal the total flow out of a node.



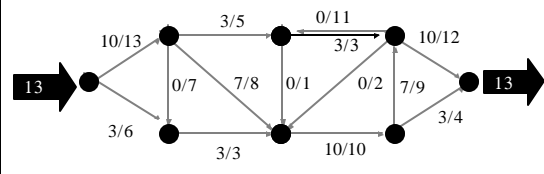
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MAX FLOW?

1. **The flow along an edge cannot exceed its capacity**
2. The total flow into a node must equal the total flow out of a node.



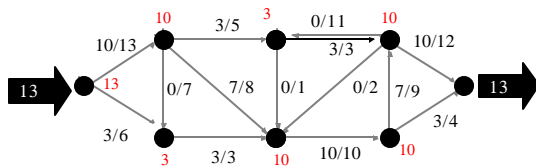
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MAX FLOW?

1. The flow along an edge cannot exceed its capacity
2. **The total flow into a node must equal the total flow out of a node.**



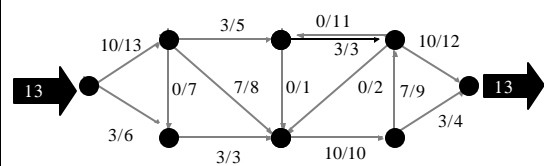
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Feasible Flow! But is it a Max Flow?

1. The flow along an edge cannot exceed its capacity
2. The total flow into a node must equal the total flow out of a node.



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Cut of a Flow Network

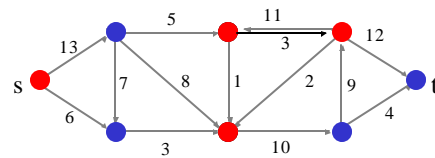
- Cut (of a flow network) is a partition of the vertices into two sets R and B such that $s \in R$ and $t \in B$

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Example of a cut



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Capacity of a cut

- The capacity of a cut (R, B) of a network is the sum of the capacities of the edges that go from a R vertex to a B vertex.

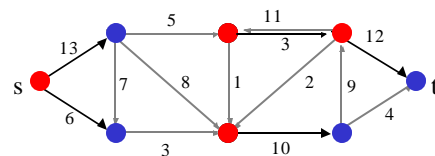
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Example of cut capacity

(R, B) has capacity 41



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Max Flow-Min Cut Theorem

- The maximum flow of a network is equal to the capacity of the minimum capacity cut in the network.

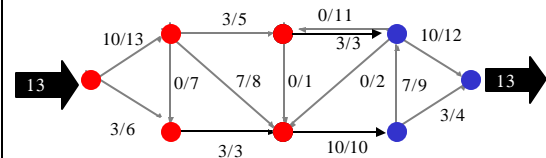
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Feasible Flow! But is it a Max Flow?

Here is a matching cut! But is it a min-cut?



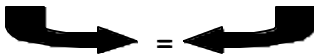
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Max Flow-Min Cut Theorem

- The maximum flow of a network is equal to the minimum capacity of any cut in the network.
- $\text{Flow} \leq \text{Max flow} = \text{Min cut} \leq \text{Cut}$



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Network Flow What is ahead?

- Algorithms for finding maximum flow in a network
- Reducing problems to the max flow problem

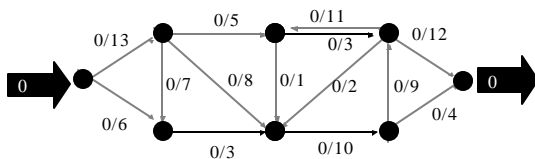
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Greedy approach (doesn't quite work)

- Find an augmenting path



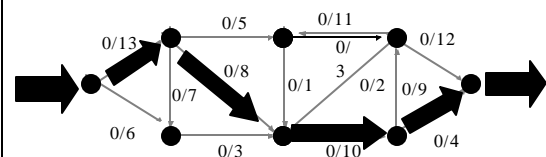
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Augmenting path

Find an augment path: An $s \Rightarrow t$ path in which the capacity of each edge exceeds its flow.



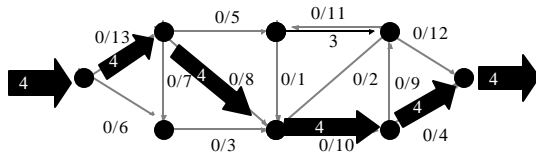
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Augmenting path

1. Find an augment path
2. Push as much flow through the path as possible.



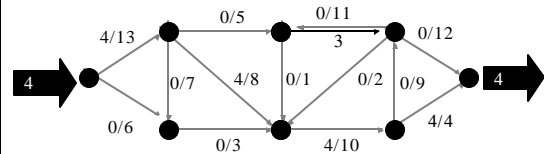
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Repeat

1. Find an augment path
2. Push as much flow through the path as possible.



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Greedy: Augmenting paths

- Does it work?
(i.e. does it lead to a max flow?)

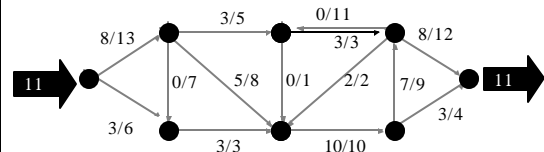
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Augmenting Path in network doesn't always work!

We can get stuck! What should we do?



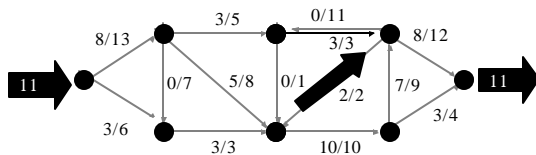
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Augmenting Path in network doesn't always work!

What should we do? Reroute!



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One more idea

Given a network and a flow:

The flow across a cut (R, B) is

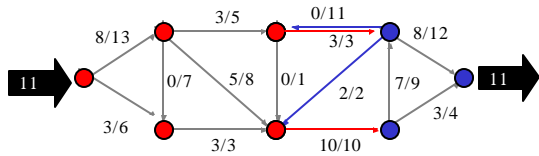
$$\sum f(\langle u, v \rangle) - \sum f(\langle v, u \rangle)$$

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Flow across (R,B)

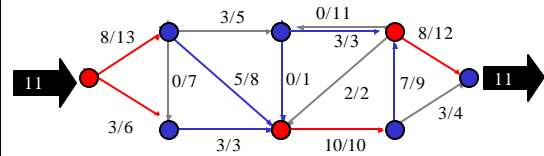


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Flow across (R,B)



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Flow across a cut

- Conservation of flow
- ↓
- The flow across any cut equals the network flow

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Flow across a cut

- Conservation of flow + max flow/min cut theorem
- ↓
- If flow is max then for minimum capacity cut (R,B): $\sum f(<v,u>) = 0$

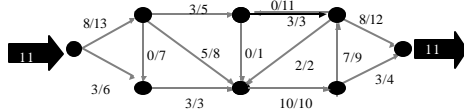
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Build residual graph

1. Add e if $\text{flow}(e) < \text{capacity}(e)$



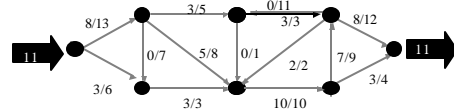
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Build residual graph

2. Add reverse(e) if $\text{flow}(e) > 0$

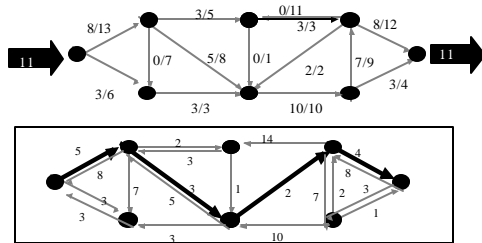


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Find $s \rightarrow t$ path in the residual graph

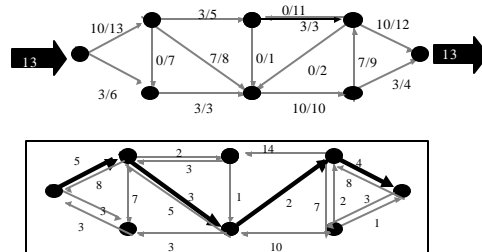


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Augment the $s \rightarrow t$ path in the network

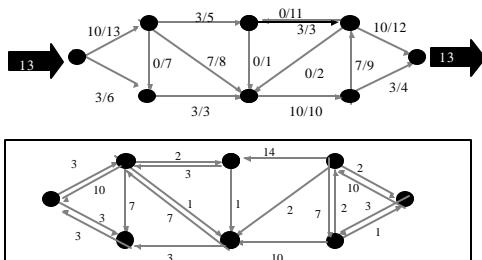


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And then ...
Repeat until residual graph has no $s \rightarrow t$ path



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Ford-Fulkerson

- Build residual graph
- If $s \rightarrow t$ path does not exist then return current flow
- Find $s \rightarrow t$ path
- Augment flow in network
- Repeat

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Augmenting Path Method (Ford-Fulkerson)

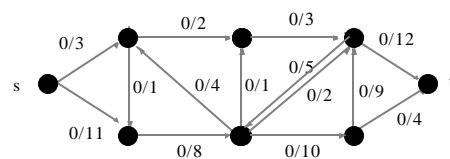
- Is it correct?
- Is it efficient?

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Exercise

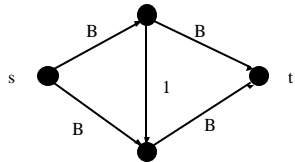


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Problems: How many augmentations could this take?

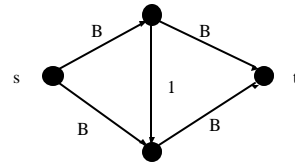


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B Augmentations – What is size of input?

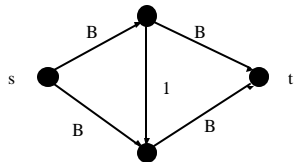


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B Augmentations
Input has size $n + m \lg(B)$



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Edmonds-Karp

- Choose shortest $s \rightarrow t$ path in residual graph to augment



- Total augmentations is $O(mn)$

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Edmonds-Karp

- Build residual graph
- If $s \rightarrow t$ path does not exist then return current flow
- Find shortest $s \rightarrow t$ path
- Augment flow in network
- Repeat

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