CS140: Exam 3 Name ______________________________

12/6/01

This exam consists of 4 problems and 21 pages. Before you begin the exam make sure your copy is complete. You have four hours to complete this exam. The exam must be turned in by Sunday morning at 10AM.

Your answers will be judged for clarity, conciseness, and correctness. The space provided should be adequate for your solutions.

You may use a one page cheat sheet but no other texts, notes, web pages, etc. during this exam. You may not discuss the exam with anyone.

The next page of the exam gives the definition of several NP-hard problems you may find useful.

Good luck!

Write your name above. Write the day and time that you start and stop the exam in the space provided.

Start Day/Time ______________________________

End Day/Time ______________________________

Here are some NP-hard problems you may want to use in completing this exam.

1. Vertex Cover

   Input: A graph $G = (V, E)$ and an integer K

   Question: Is there a subset $W \subseteq V$ containing K or fewer vertices such that every edge of G is incident to some vertex in W?

2. Dominating Set

   Input: A graph $G = (V, E)$ and an integer K

   Question: Is there a subset $W \subseteq V$ containing K or fewer vertices such that every vertex of G is either in W or adjacent to a vertex in W?

3. 3-Colorability

   Input: A graph $G = (V, E)$

   Question: Can the vertices of G be colored *red, green,* and *blue* so that no adjacent vertices have the same color?

4. Hamiltonian Path

   Input: A graph $G = (V, E)$

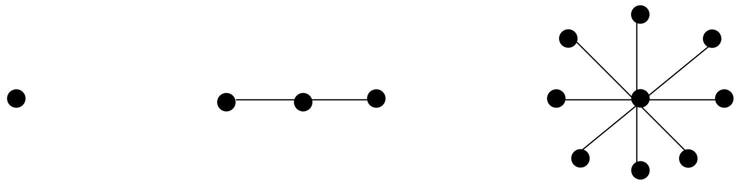
   Question: Does G contain a simple path that includes every vertex?

5. 3SAT

   Input: A boolean formula $\mathcal{F}$ in conjunctive normal form in which every clause contains exactly 3 literals.

   Question: Is $\mathcal{F}$ satisfiable?

1. A star on n vertices is a tree in which no more than one node has degree greater than one. Following are examples of stars for $n = 1, 3$, and 9.

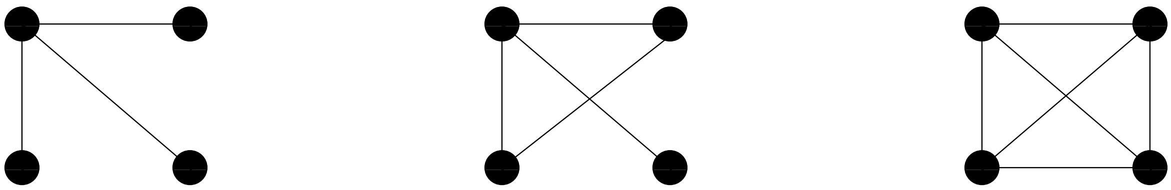


The Star Partition problem (SP) is defined as follows:

Input: Connected graph $G = (V, E)$ and integer K.

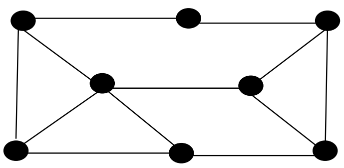
Question: Can the vertices of G be partitioned into K or fewer sets $V_1, \ldots, V_k$ so that the induced subgraph on each V_i contains a star on $||V_i||$ nodes?

For example, the following graphs contain a star on 4 nodes.



(a) State whether the following is a yes or a no instances of SP.

G=



K=2

(b) Prove that SP is in NP.

(c) Consider the following search variant of SP.

Input: Connected graph $G = (V, E)$

Output: A partition $V_1, \ldots, V_k$ of V such that k is as small as possible and the subgraph of G induced by each V_i contains a star on $||V_i||$ nodes.

Give a polynomial-time reduction from this search problem to the decision problem.

(d) Show that SP is NP-hard by giving a polynomial time reduction from one of the NP-complete problems described at the beginning of this exam. More specifically, answer the following questions.

i. Which NP-hard problem will you use?

ii. Given an input X of the selected problem, how do you construct the input G, K for SP? Show that the construction can be done in polynomial-time.

iii. Suppose the input X of the selected problem is a yes-instance. Prove that the instance G, K of SP created by your reduction is also a yes-instance.

iv. Suppose that the instance G, K of SP created by your reduction is a yes-instance. Prove that the instance X of the selected problem is also a yes-instance.

2. The Hitting String (HS) problem is defined as follows.

   Input: A set of strings C over the alphabet $\{0, 1, 2\}$, all having the same length n.

   Question: Is there a string x over the alphabet $\{0, 1\}$ such that x has length n and for each $y \in C$ there is some i such that the ith symbol of x and y agree?

   (a) The following is a yes-instance of HS. Prove it by giving an appropriate string x over $\{0, 1\}$. (Note that x cannot contain the symbol 2.)

   $$C = \{0112, 1102, 0001, 2020, 2101\}$$

   (b) Give a no-instance of HS.

   (c) Show that HS is in NP.

(d) Show that HS is NP-hard by giving a reduction from 3SAT. More specifically, answer the following questions.

   i. Given an input $\mathcal{F}$ of 3SAT, how do you construct the input C for HS? Show that the construction can be done in polynomial-time.

ii. Suppose the input $\mathcal{F}$ is satisfiable. Prove that the instance C of HS created by your reduction is also a yes-instance.

iii. Suppose that the instance C of HS created by your reduction is a yes-instance. Prove that the formula $\mathcal{F}$ is satisfiable.

3. The Maximum Leaf Spanning Tree (MLST) problem is defined as follows.

   Input: A connected graph $G = (V, E)$ and an integer K

   Question: Is there a spanning tree of G in which K or more vertices have degree 1?

   (a) Give a yes-instance of MLST.

   (b) Give a no-instance of MLST.

   (c) Show that MLST is in NP.

(d) Consider the following search problem.

Input: Connected graph $G = (V, E)$

Output: A spanning tree of G with a maximum number of degree 1 vertices.

Give a polynomial time reduction from the search problem to the decision problem.

(e) Show that MLST is NP-hard by giving a polynomial-time reduction from Dominating Set. More specifically, answer the following questions.

   i. Given an input G, K of DS, how do you construct the input G', K' for the MLST problem? Show that the reduction can be done in polynomial-time.

ii. Suppose G, K is a yes-instance of DS. Prove that the instance G', K' of SP created by your reduction is also a yes-instance.

iii. Suppose that the instance $G'K'$ of MLST created by your reduction is a yes-instance. Prove that the instance G, K of DS is also a yes-instance.

4. The Max Cut (MC) problem is defined as follows.

   Input: A graph $G = (V, E)$ with positive (integer) edge weights and an integer K.

   Question: Is there a cut of G such that the total weight of the edges spanning the cut is at least K?

   (Recall that a cut of a graph is a partition of its vertices into two sets S_1 and S_2. An edge spans the cut if it has one endpoint in S_1 and the other in S_2.)

   (a) Give a yes-instance of MC.

   (b) Give a no-instance of MC.

   (c) Prove that MC is in NP.

(d) This problem is NP-hard and remains so even if we restrict the edges weights to be either 1 or m, where $||E|| = m > 1$. Prove that this restricted verion of MC is NP-hard by giving a polynomial-time reduction from 3SAT. More specifically, answer the following questions.

i. Suppose that $G = (V, E), \quad K$ is an instance of MC where $K = \ell m + j$ for some $\ell \geq 0$ and $0 \leq j < m$. Give a necessary condition on the edge weights of G for it to be a yes-instance. (This question is intended to be a hint to help you in the reduction so make sure you figure it out before you go on.)

ii. Design a “truth-setting” gadget. (Hint: Rather than calling the partition sets S_1 and S_2 you may want to call them T and F!)

iii. Design a “clause-testing” gadget. (Hint: Think triangles.)

iv. Complete the reduction and show that it can be done in polynomial-time.

v. Suppose $\mathcal{F}$ is a satisfiable 3SAT formula. Show that the instance G, K of MC created by your reduction is a yes-instance.

vi. Suppose the instance G, K of MC created by your reduction on input $\mathcal{F}$ is a yes-instance. Show that $\mathcal{F}$ is satisfiable.