# CS140: Algorithms

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Lecture 1



## Last class

The two important questions we consider in CS140:

- Is the computational procedure correct?
- **Is the algorithm fast?**

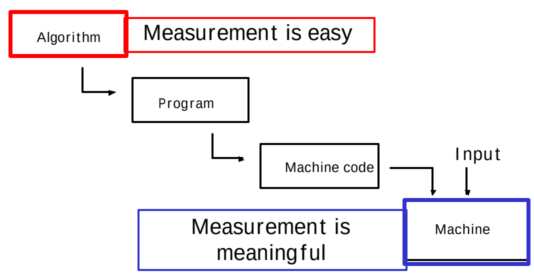


## How do we measure speed?

- **What to measure**
- Big-O notation/rate of growth
- Loop counting
- Series
- Recurrence relations



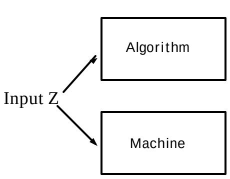
## Running Time
## Where to measure?






## A useful assumption

$T_A$ and $T_M$ differ by no more than a multiplicative constant



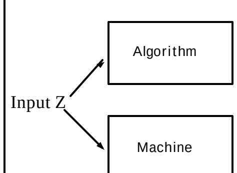

$T_A(Z)$: number of steps of algorithm on input Z

$T_M(Z)$: number of steps of machine on input Z



## More formally

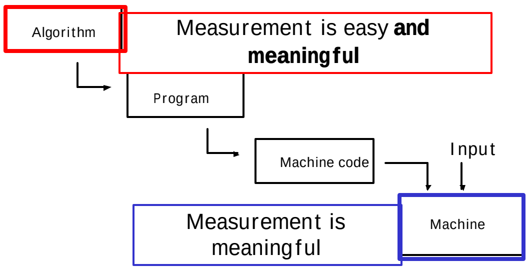
There is some constant c such that for any input Z $T_A(Z)/c \leq T_M(Z) \leq cT_A(Z)$




$T_A(Z)$: number of steps of algorithm on input Z

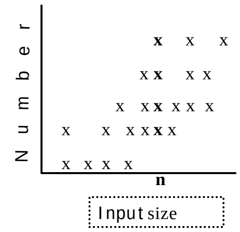
$T_M(Z)$: number of steps of machine on input Z

## Running Time
## Where to measure?





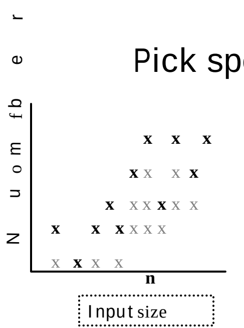

## Running Time:
## What to measure?




- Run time depends on input size
- Run time can vary on different inputs of size n.



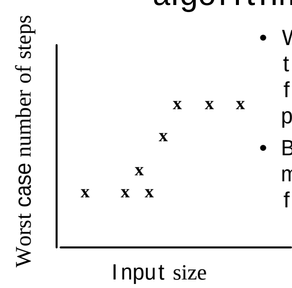
## Pick special case




- Run time depends on input size
- Run time can vary on different inputs of size n.
- Choose case:
  - Worst case (show in bold)
  - Best case
  - Average case
  - Etc.



## Worst case performance of algorithm A




- We can compute this function at a finite number of points .
- Better yet, we can model this function for all input sizes.



## A general problem ...

- Question: How can we give a succinct description of an arbitrary function?
- Answer: Big-O notation.

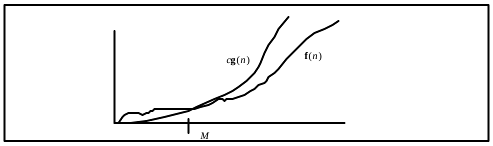


## Today

- What to measure
- **Big-O notation/rate of growth**
- Loop counting
- Series
- Recurrence relations

## Upper Bounds

- $\mathbf{f}: N \rightarrow N$ and $\mathbf{g}: N \rightarrow N$ are positive-valued, monotonically increasing functions.
- $O(\mathbf{g}(n)) = \{\mathbf{f}(n)$: there are constants $c$ and $M$ such that $\mathbf{f}(n) \leq c\,\mathbf{g}(n)$ for all $n \geq M\}$






## Proving f(n)=O(g(n))

Consider h(n)=f(n)/g(n) as n goes to infinity

- h(n) converges
- h(n) diverges
  - Unbounded
  - Bounded



## Some useful observations about Big-O

- If f(n)/g(n) is bounded then f(n) ___O(g(n)).
- If f(n)/g(n) is unbounded then f(n)___O(g(n)).



## Logarithms

For which pairs f(n), g(n) is f(n)=O(g(n))?

lg n

$\log n^2$

$\lg^2 n$

lg 10000n



## Example

$\lim_{n\to\infty} \lg n / \log n^2 = (\lg 10)/2$

( Useful observation: $\log n^2 = (2/\lg 10) \lg n$)



## Limits

1. $\lim_{n\to\infty} \log n^2 \;/\; \lg n$
2. $\lim_{n\to\infty} \lg n \;/\; \lg^2 n$
3. $\lim_{n\to\infty} \lg^2 n \;/\; \lg n$
4. $\lim_{n\to\infty} \lg n \;/\; \lg 10000\, n$

## Polynomials

For which pairs f(n), g(n) is $f(n)=O(g(n))$?

$n$

$n^2$

$1000n^2 + n$



## Exponentials

For which pairs f(n),g(n) is $f(n)=O(g(n))$?

$2^n$

$3^n$

$2^{(n^2)}$

$(2^n)^2$



## Some rules of thumb

- Polylogs are slower growing than polynomials

  For any k,j > 0:

  $\log^j n = O(n^k)$ and $n^k \neq O(\log^j n)$
- Polynomials are slower growing than exponentials

  For any k>0 and r>1:

  $n^k = O(r^n)$ and $r^n \neq O(n^k)$



## L'hospital's rule

- $\lim_{\rightarrow\infty} \log^j n \;/\; n^k = 0$
- $n^k \;/\; \log^j n$ diverges as n goes to infinity



## Polynomially bounded functions

f(n) is polynomially bounded if there is a constant k such that $f(n) = O(n^k)$



## Logs, Polys, and Exps

Which of the following functions are polynomially bounded?

$\log n$

$n^3$

$2^n$

## Other functions

- Factorial: $n! = n\,(n-1)!$, $0! = 1$
- Tower of 2s: $2\uparrow\uparrow n = 2^{2\uparrow\uparrow(n-1)}$, $2\uparrow\uparrow 0 = 1$
- Iterated log: $\text{Log}^*(n) = m$ such that $2\uparrow\uparrow(m-1) < n \le 2\uparrow\uparrow n$
- Ceil-ceil: $\lceil\lceil n \rceil\rceil = 2^m$ such that $m-1 < \lg n \le m$



## Logs, polys, exps, and others

Compare the rates of growth of the following functions:

$\lg n$, $n^3$, $2^n$, $n!$, $2\uparrow\uparrow n$, $\log^*(n)$, $\lceil\lceil n \rceil\rceil$



## Another useful observation

- If $f(n)/g(n)$ diverges then so does $2^{f(n)}/2^{g(n)}$
- If $\lg(f(n))/\lg(g(n))$ diverges then so does $f(n)/g(n)$



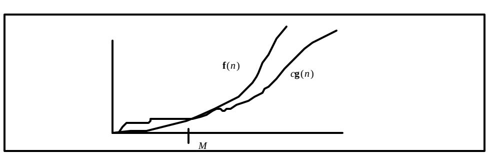
## Beyond O

| <u>real numbers</u> | <u>functions</u> |
| --- | --- |
| • $\le$ | • O |
| • $\ge$ | • $\Omega$ |
| • = | • $\Theta$ |
| • < | • o |
| • > | • $\omega$ |



## Lower Bounds

- $\mathbf{f}:N \rightarrow N$ and $\mathbf{g}:N \rightarrow N$ are positive-valued, monotonically increasing functions.
- $\Omega(\mathbf{g}(n)) = \{\mathbf{f}(n)$: there are constants $c$ and $M$ such that $\mathbf{f}(n) \ge c\,\mathbf{g}(n)$ for all $n \ge M\}$






## Definition: $\Theta$

$f(n) = \Theta(g(n))$ if the following hold:

1. $f(n) = O(g(n))$, and
2. $f(n) = \Omega(g(n))$

## Definition: little-o, little-ω

- $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$
- $f(n) = \omega(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = \infty$



## Logs, polys, exps, and others

Compare the following functions.
Which of O, Ω, Θ, o, and ω apply?

$\lg n$, $n^3$, $2^n$, $n!$, $2\uparrow\uparrow n$, $\log^*(n)$, $\lceil\lceil n\rceil\rceil$



## A slight twist…

Is $f(2n) = O(f(n))$?

1. $f(n) = 1$: Is $2n=O(n)$?
2. $f(n) = 3n$: Is $6n=O(3n)$?
3. $f(n) = n^2$: Is $4n^2=O(n^2)$?
4. $f(n) = 2^n$: Is $4^n=O(2^n)$?
5. $f(n) = n!$: Is $(2n)!=O(n1)$?



## Today

- What to measure
- **Big-O notation/rate of growth**
- Loop counting
- Series



## CS140 pragmatism

What is the asymptotic behavior of the worst-case running time of the algorithm?



## CS140 pragmatism

Big-O

What is the **asymptotic behavior** of the worst-case running time of the algorithm?

## CS140 pragmatism

What is the asymptotic behavior of the **worst-case** running time of the algorithm?

Special case input



## CS140 pragmatism

What is the asymptotic behavior of the worst-case **running time** of the algorithm?

Chosen resource



## CS140 pragmatism

What is the asymptotic behavior of the worst-case running time of the **algorithm**?

Remember our assumption



## In this class we'll say

The running time of ▲ is O($n^3$).

$\Updownarrow$

The worst case running time of ▲ is O($n^3$).

$\Updownarrow$

▲ is O($n^3$).



## Rate of growth of common functions

- Review of properties/notation
- See CLR pp 32-37 for details

**KNOW THIS STUFF**



## Today

- How should we measure the speed of an algorithm?
- Big-O notation/rate of growth
- **Loop counting**
- Series

## Types of Algorithms

- Recursive Algorithm: one that calls itself
- Purely Iterative Algorithm: one that doesn't



## Run Time Analysis

- **Iterative algorithm** ® **Loop counting**
- Recursive algorithm → Recurrence relations



## Iterative Sorting Algorithms

- Insertion-sort
- Bubble-sort
- Modified Bubble-sort



## Insertion-sort(S)

(in pseudo-code)

S is an array of n integers: S(1), S(2), …, S(n)

```
For j = 2 to n
   key = S(j)
   i = j-1
   While i > 0 and S(i) > key
      S(i+1) = S(i--)
   S(i+1) = key
```

C decrement operator



## Correctness

- Inductive proof with loop invariant:
  When the **for loop** executes for the $k^{th}$ time, S(1), S(2), …, S(k) are sorted in ascending order.



## Loop Counting: Insertion-sort(S)

| | |
|---|---|
| $\Sigma_{j=2..n}$ (1 → | For j = 2 to n |
| +1 → | key = S(j) |
| +1 → | i = j-1 |
| + $\Sigma_{i=0..j-1}$ (1 → | While i>0 & S(i)>key |
| +1) → | S(i+1) = S(i--) |
| +1) → | S(i+1) = key |

$$\boxed{\Sigma_{j=2..n}(4+\Sigma_{i=0..j-1}2) = O(n^2)}$$

## Bubble-sort(S)

```
Bubble-sort(S)
    For i=n down to 2
        For j=1 to i-1
            If S(j) >S(j+1) then swap(S(j), S(j+1)
            )
    Return
```



## Correctness

- Inductive Proof with loop invariant:
  When the i-loop completes its $k^{th}$ execution,
  - S(n-k+1), S(n-k+2), ..., S(n) is sorted in ascending order, and
  - the max(S(1), ..., S(n-k)) ≤ S(n-k+1).



## Does Bubble-sort do too much work?

1, 3, 2, 4, 5
1, 2, 3, 4, 5
1, 2, 3, 4, 5
1, 2, 3, 4, 5
1, 2, 3, 4, 4

- Repeat on smaller list



## Modified Bubble-sort

```
Modified-Bubble-sort(S)
    SWAP=T
    For i=n down to 2
        If SWAP=F then return
        SWAP=F
        For j=1 to i-1
            If S(j)>S(j+1) then swap(S(j),S(j+1)) and
             set SWAP=T
    Return
```



## Example

1, 3, 2, 4, 5
1, 2, 3, 4, 5
1, 2, 3, 4, 5

- Repeat on smaller list unless no swaps are made



## Loop counting: M-Bubble-sort

| | Modified-Bubble-sort(S) |
|---|---|
| $1 +$ | SWAP=T |
| $\Sigma_{i=2..n}($ $1+$ | For i=n down to 2 |
| $1+$ | If SWAP=F then return |
| $1+$ | Flag=F |
| $\Sigma_{j=1..i-1}($ $1+$ | For j=1 to i-1 |
| $3)$ | If $s_j > s_{j+1}$ then swap($s_j$, $s_{j+1}$) and set SWAP=T |
| | Return |

$$\boxed{1+\Sigma_{i=2\ldots n}(3+\Sigma_{j=1\ldots i-1} 4)) = O(n^2)}$$

## Summation

$\Sigma_{i=2...n} \Sigma_{j=1...i-1} c = c\Sigma_{i=2...n}(i-1)$

$= c(\Sigma_{i=1...n}(i-1)) - c$

$= c(\Sigma_{i=1...n} i - \Sigma_{i=1...n} 1) - c$

$= c(n(n+1)/2 - n) - c$

$= \mathbf{O(n^2)}$

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## Series

- A series is a summation of terms
- Common series
  - Arithmetic series: $1+2+...+n$
  - Geometric series: $1+a+a^2+...+a^n$

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## Series Things we want to do:

- **Solve exactly**
- Bound above or below
- Prove that a solution (or bound) is correct

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## Closed form solutions to some common series

- $f(n) = 1+2+ ... + n = n(n+1)/2$
- $f(n) = 1^2 + 2^2 + ... + n^2 = (2n^3 + 3n^2 + n)/6$
- $f(n) = 1+a+a^2+...+a^n = n$ if $a=1$
  $= (a^{n+1}-1)/(a-1)$ else
- $f(n) = 1+a+a^2+... = 1/(1-a)$ if $0 \le a < 1$

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## Series Things we want to do:

- Solve exactly
- **Bound above or below**
- Prove that a solution (or bound) is correct

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## Upper Bounds on series

For any constant k:

$\Sigma_{i=1..n} i^k \lesssim \Sigma_{i=1..n} n^k$

$= (n^{k+1})$

$= O(n^{k+1})$

Is this a good upper bound?

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## Lower Bounds on series

For any constant k:

$$\Sigma_{i=1..n} i^k \geq \Sigma_{i=\lfloor n/2 \rfloor \dots n} i^k$$
$$\geq \Sigma_{i=\lfloor n/2 \rfloor \dots n} \lfloor n/2 \rfloor^k$$
$$\geq \lfloor n/2 \rfloor^{k+1}$$
$$= \Omega(n^{k+1})$$

So $\Sigma_{i=1..n} i^k = \Theta(n^{k+1})$



## Series
## Things we want to do:

- Solve exactly
- Bound above or below
- **Prove that a solution (or bound) is correct**



## Proving correctness

- Claim: $\Sigma_{i=1..n} i^2 = (2n^3 + 3n^2 + n)/6$
- Claim holds for n=1.
- If the claim holds for n then it holds for n+1:

$$\Sigma_{i=1..n+1} i^2 = (n+1)^2 + \Sigma_{i=1..n} i^2$$
$$= (n+1)^2 + (2n^3 + 3n^2 + n)/6$$
$$= (2(n+1)^3 + 3(n+1)^2 + (n+1))/6$$



## Run Time Analysis

- Iterative algorithm $\rightarrow$ Loop counting
- **Recursive algorithm ® Recurrence relations**
  1. **Write the recurrence relation**
  2. **Convert the recurrence relation to a series**
  3. **Solve the series**



## Sort3: A Recursive Algorithm for SIAO

```
Sort3(S)
  If ||S||≤1
      Return: S
  Else
      Return: Sort3(S\max-element(S)),max-element(S)
```



## 1. Write the recurrence relation

```
Sort3(S)
  If ||S||≤1
      Return: S
  Else
      Return: Sort3(S\max-element(S)),max-element(S)
```

Let T(n) be the running time of Sort3:

$T(1) = c_2$
$T(n) = c_1 n + T(n-1)$, n>1

## 2. Convert to series

$T(1) = c_2$
$T(n) = c_1 n + T(n-1)$ , $n>1$

$\Downarrow$

$c_2 + \Sigma_{i=2\ldots n} c_1 n$

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## 3. Solve

$c_2 + \Sigma_{i=2\ldots n} c_1 n \Rightarrow c_2 - c_1 + c_1 n(n+1)/2 = O(n^2)$

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## Recurrence Relations

Shortcuts and other tools:
- **Guess and prove**
- Master method
- Unwinding
- WORK TREES

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## Steps 2-3: Guess and Prove

- $T(n)=c_1 n + T(n-1)$, $T(1)=c_2$
- Guess: $T(n) = O(n^2)$
- Prove: We need to show that there exists constants c and M such that $T(n) \leq cn^2$ for all $n \geq M$

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## Guess and Prove cont.

- $T(1) \leq c$, $\boxed{\text{provided } c \geq c_1}$
- Suppose $T(n-1) \leq c(n-1)^2$.

$$\begin{aligned} T(n) &= c_2 n + T(n-1) \\ &\leq c_2 n + c(n-1)^2 \\ &= c_2 n + c(n^2 - 2n + 1) \\ &= cn^2 - (2c - c_2)n + c \\ &\leq cn^2, \ \boxed{\text{provided } c \geq c_2 \text{ and } n \geq 1} \end{aligned}$$

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## Guess and Prove cont.

- $T(n) \leq cn^2$ for all $n \geq 1$, where $c=\max(c_1,c_2)$

$\Downarrow$

- $T(n) = O(n^2)$

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## Guess and Prove cont.

- What if you guess is wrong?
- You'll reach a contradiction in the proof step



## Recurrence Relations

Shortcuts and other tools:
- Guess and prove
- **Master method**
- Unwinding
- WORK TREES



## Steps 2-3: Master Theorem

- Read the book

Warning – only works for certain types of recurrence relations



## Recurrence Relations

Shortcuts and other tools:
- Guess and prove
- Master method
- **Unwinding**
- WORK TREES



## Step 2: Unwinding

$$
\begin{aligned}
T(n) &\le c_2 n + T(n-1) \\
&\le c_2 n + c_2(n-1) + T(n-2) \\
&\le c_2 n + c_2(n-1) + c_2(n-2) + T(n-3)
\end{aligned}
$$

.

.

.

$T(n) \le c_1 + \Sigma_{i=2..n}\, c_2 i$



## Recurrence Relations

Shortcuts and other tools:
- Guess and prove
- Master method
- Unwinding
- **WORK TREES**

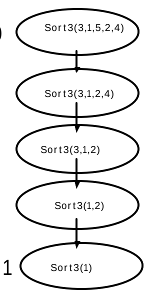
## Work Tree

Graphical representation of the "work" done by a recursive algorithm



## Example: Sort3(3,1,5,2,4)

Level 0

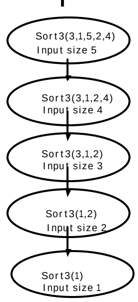



Level n-1

Tree Nodes = Recursive calls



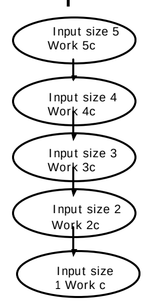
## Example: Sort3(3,1,5,2,4)




Input size at level i is n-i



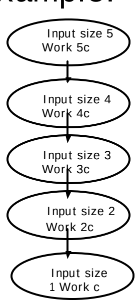
## Example: Sort3(3,1,5,2,4)




Work done at level i is c(n-i)



## Example: Sort3(3,1,5,2,4)




Total work:

$\Sigma_{i=0...n-1}\, c(n-i) = cn(n+1)/2 = O(n^2)$



## Next time

- Work trees continued
- MergeSort
- A little probability theory
- QuickSort