CS140: Algorithms

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Lecture 5

Today

- Lower bounds
  - **Counting arguments**
  - Ad hoc arguments
  - Adversary arguments

Run Time Bounds

Worst Case running times of classic sorting algorithms:
- Bubble-sort: $\Theta(n^2)$
- Insertion-sort: $\Theta(n^2)$
- Merge-sort $\Theta(n \log(n))$
- Heap-sort $\Theta(n \log(n))$
- Quick-sort $\Theta(n^2)$

Comparison-based sorting

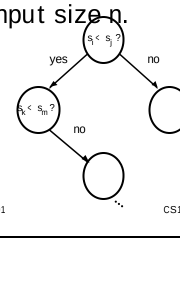
A comparison-based sorting algorithm is one that doesn't need to read the input, provided it is given the size of the input and a comparison oracle.

Lower Bound for Sorting

Theorem: Any comparison-based sorting algorithms has a worst-case running time that is $\Omega(n \log(n))$.

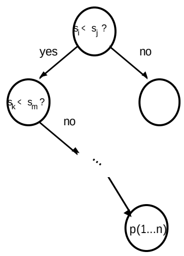
Proof of Theorem

A decision tree describes the queries of a comparison-based algorithm on input size n.



A root to leaf path represents the sequence of queries for a particular input.

Proof of Theorem cont.



Each leaf corresponds to the permutation that sorts the input.

Proof cont.

- There must be at least n! leaves.
- A binary tree with n! leaves has a path with length at least log(n!).
- By Stirling's approximation

$$\log(n!)=\Omega(n \log(n)).$$

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FIND-MIN

- How many comparisons does it take to find the minimum in a set of integers?
- Answer: n-1

FIND-MIN

In worst case

- How many comparisons does it take to find the minimum in a set of integers?
- Answer: n-1

Upper Bound for FIND-MIN

Upper Bound Theorem: Finding the minimum in a set of n integers requires no more than n-1 comparisons.

Proof: Give algorithm

Lower Bound for FIND-MIN

Lower Bound Theorem: Finding the minimum in a set of integers requires at least n-1 comparisons.

Proof of Lower Bound:

- Consider an algorithm ***A*** on input of size ***n***.
- Let G be a graph with a vertex for each input integer. Initially G has no edges. When ***A*** compares two input values, we'll add an edge between the corresponding vertices of G.
- ***A*** cannot conclude until G has ________ edges.
- Thus ***A*** cannot conclude until it has made ______ comparisons.

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Upper Bound for FIND-MIN/MAX

- Upper Bound Theorem: Finding the minimum and maximum in a set of n integers requires no more than $\lceil 3n/2 \rceil$-2 comparisons.
- Proof: Give an algorithm

Proof of Upper Bound:

- Algorithm for even n:
  - Make n/2 pair-wise comparisons
  - Find the maximum of the winners with n/2-1 comparisons
  - Find the minimum of the losers with n/2-1 comparisons
- Algorithm for odd n:
  - Run even algorithm on first n-1 integers
  - Compare the min and max to the last integer

Lower Bound for FIND-MIN/MAX

- Lower Bound Theorem: Finding the minimum and maximum in a set of n integers requires at least $\lceil 3n/2 \rceil$ - 2 comparisons.
- Proof: Adversary argument

Example of an adversary

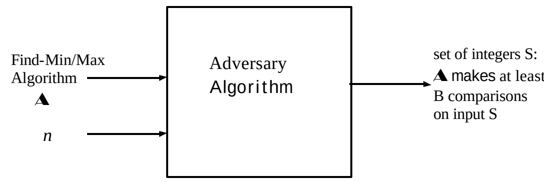
You pick a number y between 1 and 100

I have to guess y by posing queries of the form "Is it x?"

You answer "yes, x=y" or "no, y<x" or "no, y>x"

- How many queries can you force me to make?
- Prove it!

Adversary Argument to prove bound B
(for FIND MIN/MAX)



FIND-MIN/MAX Adversary - Accounting

- Adversary = interactive comparison oracle
- Accounting scheme: For x in S

$b_{MAX}(x)=$ 1 if the algorithm can rule out x as the largest integer; 0 otherwise

$b_{MIN}(x)=$ 1 if the algorithm can rule out x as the smallest integer; 0 otherwise

FIND-MIN/MAX Adversary- Strategy

- On query "Is x<y ?"
- Answer consistently with previous answers
- If yes and no both consistent then answer so as to minimize the changes in b_{MAX} and b_{MIN} variables

FIND-MIN/MAX Adversary - Analysis

Consider a query "Is x<y?"

- If NO: $b_{MIN}(x) \to 1$ and $b_{MAX}(y) \to 1$
- If YES: $b_{MAX}(x) \to 1$ and $b_{MIN}(y) \to 1$

Proof of Lower Bound:

- Claim: At most $\lfloor n/2 \rfloor$ queries can result in the change of two $b_{MIN/MAX}$ variables
- Claim: 2n-2 changes must occur before the algorithm concludes

$\Rightarrow \lfloor n/2 \rfloor + (2n-2) - 2(\lfloor n/2 \rfloor$ queries are necessary

Find-gap

- Input: S: $x_1, x_2, \ldots, x_n$ is a list of distinct integers sorted in ascending order.
- Question: Is there an index *i* such that $x_i+1<x_{i+1}$
- How many elements of S have to be read (in worst case) in order to answer?

Exercise

- What is a good adversary strategy?
- What is a good algorithm strategy

Double 0's

- Input: B: $b_1 \ldots b_n$ n-bit vector of 0/1's
- Question: Are there two adjacent 0's?
- How many bits of B have to be read (in worst case) in order to answer?

Exercise

- What is a good adversary strategy?
- What is a good algorithm strategy

Upper Bound

Claim: Double 0's can be solved with f(n) queries where:

$f(n) = n-1$ if $n \equiv 1 \bmod 3$

$= n$ otherwise

Lower Bound

Double 0's cannot be solved with fewer than g(n) queries where:

$G(n) = n-1$ if $n \equiv 1 \bmod 3$

$= n$ otherwise