CS140: Algorithms

Z Sweedyk
Lecture 9

Outline

- Reductions to network flow
- Reductions between search and decision problems

Reductions to Network Flow Problem

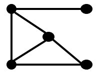
- **Bipartite Matching µ Network Flow**
- The Gee-ball Problem $\propto$ Network Flow

Matching

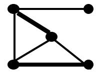
- Let G=(V,E) be a graph.
- $E' \subseteq E$ is a matching if every vertex of V is incident to at most one edge of E'.

Matching Example

G



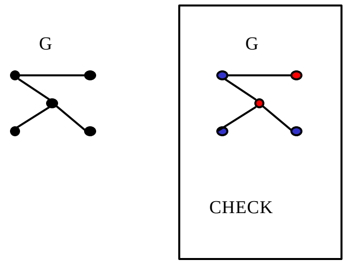
E'



Bipartite Graph

- Let G=(V,E) be a graph.
- G is bipartite if V can be partitioned into V_1 and V_2 such that no pair of vertices in V_i (i=1,2) have an edge.

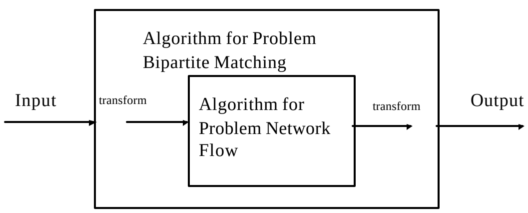
Bipartite Example



Bipartite Matching

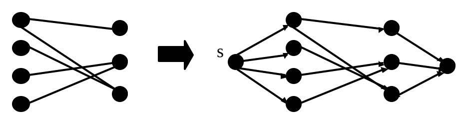
- Input: Bipartite graph G
- Output: A largest matching of G

Bipartite Matching $\propto$ Network Flow



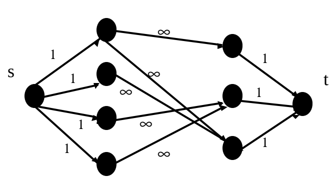
Bipartite Matching $\propto$ Network Flow

- Transform Input – Step 1

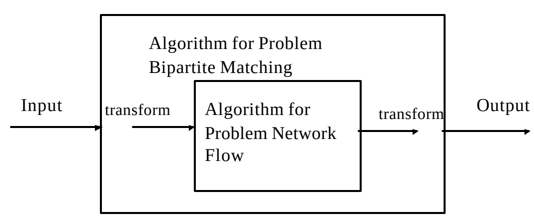


Bipartite Matching $\propto$ Network Flow

- Transform Input – Step 2

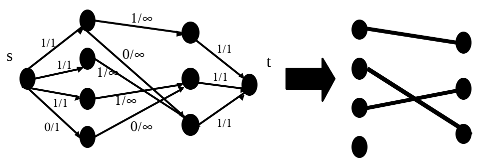


Bipartite Matching $\propto$ Network Flow



Bipartite Matching ∝ Network Flow

- Transform output



Reduction

- Is it correct?
- Is it efficient?

Integrality theorem

- If the capacities in a network are integral, then the max flow can be achieved with integral flows on each edge.

- Further the Ford-Fulkerson method yields an integral solution.

Proof of correctness

There is a 1-1 correspondence between 0/1 flows in the network and matchings in the input graph.

Reduction

- Is it correct?
- **Is it efficient?**

  $T_{BM}(n)=cn+T_{FF}(n+2)$

Reductions to Network Flow Problem

- Bipartite Matching ∝ Network Flow
- **The Gee-ball Problem µ Network Flow**

The Gee-ball Problem

- The southwestern conference of the gee-ball league consists of n+1 teams. Team n+1 is from HMC.
- We want to know whether it is possible for HMC to win more games this season than any other team in the conference.
- No ties allowed.

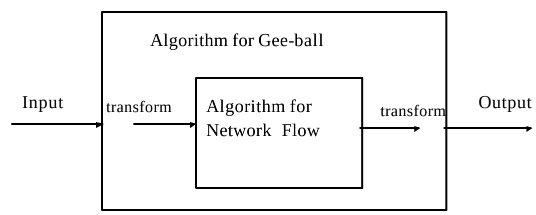
Example

- The teams are Pitzer, CMC, Pomona, and HMC
- Games won so far:
  - Pitzer 4, CMC 3, Pomona 2, HMC 2
- Games to play:
  - 1 game: Pitzer vs. HMC
  - 2 games: Pomona vs. HMC

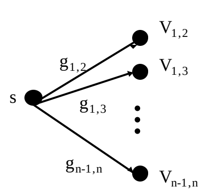
The Gee-ball Problem

- Teams $t_1, t_2, \ldots, t_n, t_{n+1}$
- So far this year team i has won w_i games.
- Teams i and j will play each other $g_{i,j}$ more times this season ($g_{i,j}=g_{j,i}$).

Gee-ball ∝ Network Flow

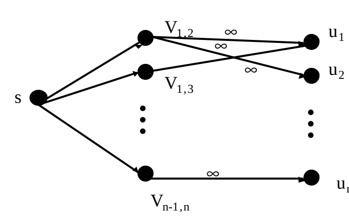


Transform Input



1. Create a source s.
2. Create vertex $v_{i,j}$ for $1 \leq i < j \leq n$
3. Create edge from s to $v_{i,j}$ with capacity $g_{i,j}$

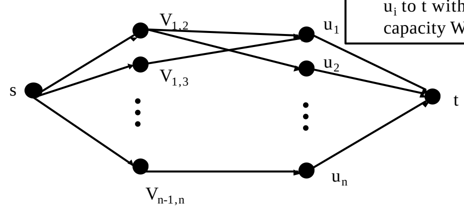
Transform Input



4. Create vertices u_i, $1 \leq i \leq n$
5. Create edge from $v_{i,j}$ to u_i and u_j with infinite capacity.

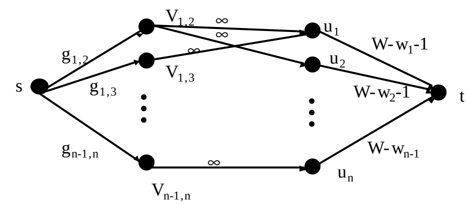
Transform Input

5. Create a sink t.
6. Create edge from u_i to t with capacity W_i.

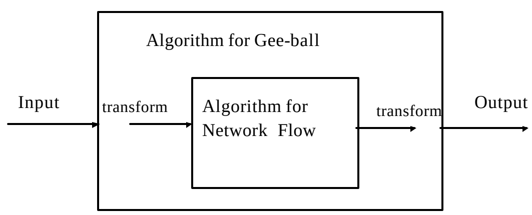


Transform Input

Let $W = w_{n+1} + \Sigma_{i=1...n}\, g_{n+1,i}$



Gee-ball ∝ Network Flow



Transform Output

There is a way for HMC to win the season if and only if ___________________________.

Decision Problems

Decision problems are computational problems for which the output is “yes” or “no”

Search: Vertex Cover

- Input: Graph G
- Output: A smallest subset of vertices W such that every edge of G is incident to some vertex in W.

Decision: Vertex Cover

- Input: Graph G and integer K
- Question: Is there a subset W of K or fewer vertices of G such that every edge is incident to some vertex in W?

Polynomial Time Reductions

- Decision-VC $\propto_P$ Search-VC
- Search-VC $\propto_P$ Decision-VC

Independent Set

- Input: Graph G and integer K
- Question: Is there a subset W of K or more vertices of G such that no two vertices of W are adjacent?

- Does Search-IS $\propto_P$ Decision-IS?

3-Colorability

- Input: Graph G
- Question: Can the vertices of G be colored with Red, Green, and Blue so that no adjacent vertices have the same color?

- Does Search-3Col $\propto_P$ Decision-3Col?