# CS140: Algorithms

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Lecture 11

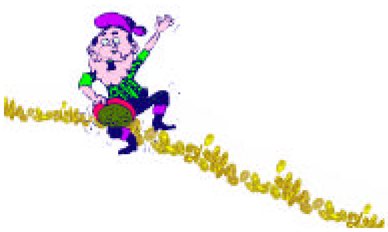


## Algorithm Design Techniques

- Induction (or Self-Reduction)
- Reduction
- Divide and Conquer (special case of Self-Reduction)
- **Greedy**



## Greedy Paradigm
## Get what you can NOW!





## But sometimes it's better to look around!



## But sometimes it isn't …



## I hate gas stations!

- I'm driving cross country and my route is fixed.
- My map tells me exactly where every gas station along the route is located.
- I want to minimize the number of times I stop for gas…
- … without running out!

## Greedy

- First stop
  - I'll stop at the farthest gas station I can get to without running out.
- Then repeat



## Greedy is Optimal!

- Can the optimal make a first stop that is later?



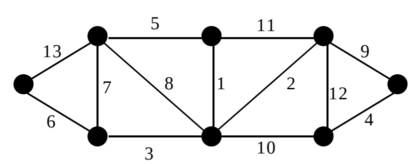
## Minimum Spanning Tree

- Input: Weighted graph G
- Output: Minimum weight spanning tree of G



## Weighted Graph

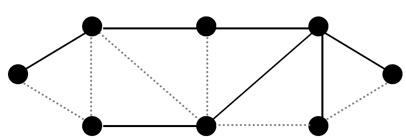
- G=(V,E) is a connected, weighted graph with n vertices and m edges.






## Spanning Tree

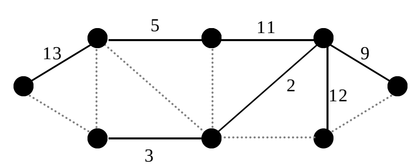
- A spanning tree of G is a connected, acyclic subgraph with vertex set V.





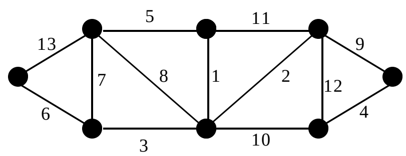
## Weight of Spanning Tree

- The weight of spanning tree of G is the sum of the weights of its edges.



## Minimum Spanning Tree

- A minimum spanning tree of G is one with smallest possible weight.
- Find an MST of the following graph:






## Prim's Algorithm

Choose a vertex $w \varepsilon V$

$F=\{w\}$

While $V-F \neq \phi$

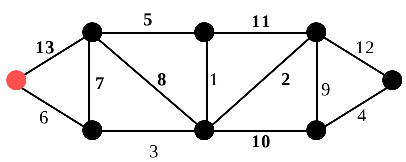
Let e be a minimum weight edge that emerges from F

$F=F+\{e\}$



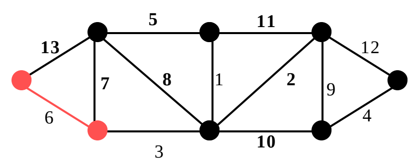
## Prim's example

Start with red vertex



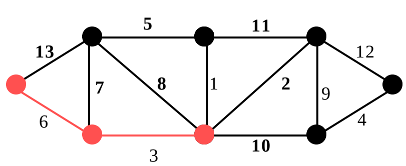



## Prim's example



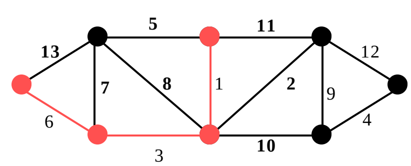



## Prim's example

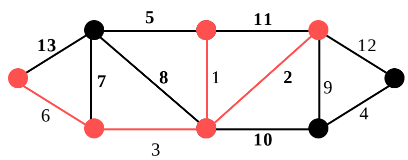





## Prim's example



## Prim's example

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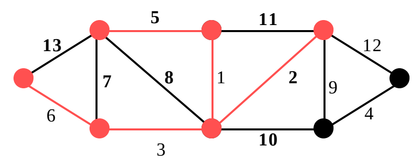
## Prim's example

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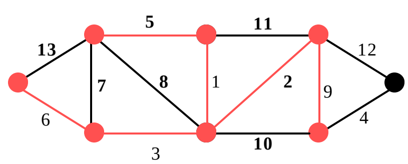
## Prim's example

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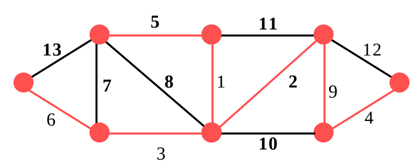
## Prim's example

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## Prim's Algorithm

- Is it correct?
- Is it efficient?

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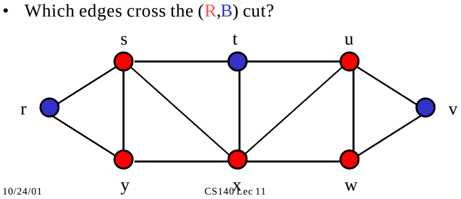
## Cut

- A **cut** is a partition of the vertices of G into two sets (R,B).
- An edge e crosses the cut if it has an endpoint in each set of the cut.
- Which edges cross the (R,B) cut?

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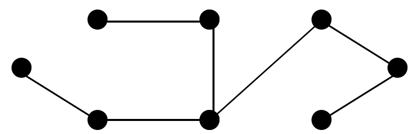
## Cut Theorem

- Let (R,B) be a cut of graph G and let e be a minimum weight edge crossing the cut.
- Then e is in some minimum spanning tree of T.
- If e is the least weight edge spanning the cut then it is in every minimum spanning tree of T.



## Tree Facts

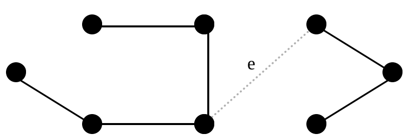
- A tree on n nodes has n -1 edges.





## Tree Facts

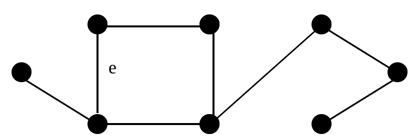
- If e is an edge of T then T-{e} is a forest consisting of two trees.






## Tree Facts

- If e is an edge of G but not of T then T+{e} contains exactly one cycle.






## Tree Facts

1. A tree on n nodes has n -1 edges.
2. If e is an edge of T then T-{e} is a forest consisting of two trees.
3. If e is an edge of G but not of T then T+{e} contains exactly one cycle.



## Proof: Cut Theorem

- Let (R,B) be a cut of graph G and let e be a minimum weight edge crossing the cut.
- Let T be a minimum spanning tree of G.
- If e is in T we are done so suppose not.
- Consider the graph T+{e}.

## Proof: Cut Theorem

- Consider the graph T+{e}.
  - By our tree facts this graph has exactly one cycle and the cycle includes e.
  - Removing any edge of the cycle yields a spanning tree of G.
  - If the cycle contains an edge e' such that w(e')≥w(e) then T+{e}-{e'} is a spanning tree with weight no more than T.



## Consider the cut (R,B)

- Since e spans the cut at least one other edge of the cycle must also span the cut.
- Why?
- So what?



## Prim's Algorithm

- Is it correct?
  - If the edge weights are unique then it follows immediately from the cut theorem.
  - What if the edge weights are not unique?
- Is it efficient?



## Prim's Algorithm

Choose a vertex w ε V

F={w}

While V-F ≠ ϕ

  Let e be a minimum weight edge that emerges from F

  F=F+{e}



## Prim's Algorithm
## Running Time: O(n(?))

Choose a vertex w ε V

F={w}

While V-F ≠ ϕ

  Let e be a minimum weight edge that emerges from F

  F=F+{e}

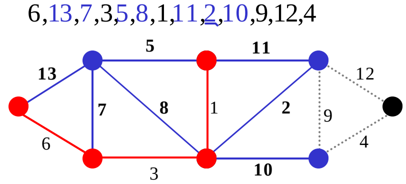
How should we implement this?



## Consider naïve approach …

- Go through edge list to find least weight edge emerging from F:

  6,13,7,3,5,8,1,11,~~2~~,10,9,12,4



## Prim's Algorithm
## Running Time: O(nm)

Choose a vertex w ε V

F={w}

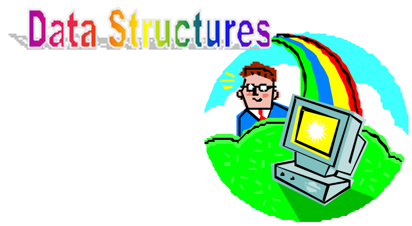
While V-F ≠ ϕ

> Let e be a minimum weight edge that emerges from F
>
> F=F+{e}

Naïve approach O(m)



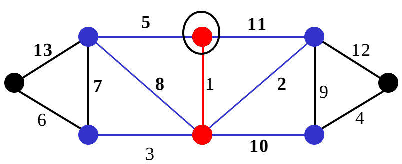
## What to do?






## Fringe vertex

v is a fringe vertex if it is in V-F and it is connected by an edge to a vertex u in F

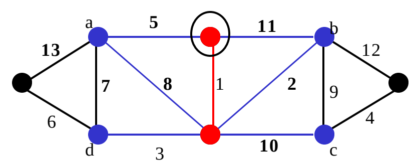





## A less naïve approach …

List of fringe vertices and for each its minimum weight edge to F

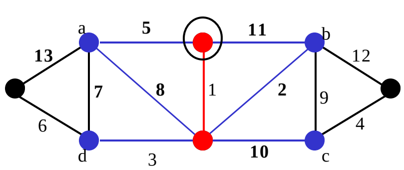
[b,2], [d,3], [a,5],[c,10]






## And even better…

- Keep the fringe vertices in a HEAP!!!!






## Prim's Algorithm
## A Better Implementation

Choose a vertex x ε V

F={x}, H=ϕ

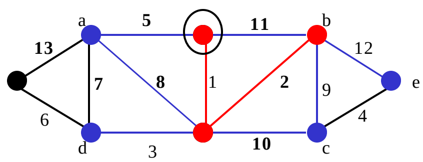
For each e=(u,x): Add record [u,e] to heap H keyed on w(e)

While H ≠ ϕ

- [u,e]=Find-min(H)
- Add u and e to F
- For each edge incident to u: Update heap

# Update heap: [b,2],[d,3],[a,5],[c,10]

- [b,2]: remove [b,2] from heap [d,3],[a,5],[c,10]
- Add new fringe vertices: [d,3],[a,5],[c,10],[e,12]
- Update edge weights: [d,3],[a,5],[c,9],[e,12]

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# Are these standard operations?

- Extract-min
- Add element to heap
- Reduce key of element in heap ?!?

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# Decrease key

- Next homework assignment: Design decrease key algorithm for heaps that runs in time O(lg(n)).

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# Prim's Algorithm Running Time

Choose a vertex $x \in V$

$F=\{x\}$, $H=\phi$

For each e=(u,x): **Insert**

While $H \neq \phi$

[u,e]=**Extract-min**(H)

Add u and e to F

For each edge incident to u: **Insert or Decrease-key** or do nothing

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# Prim's Algorithm Running Time

- Heap operations across algorithm:
  - n Extract-mins O(lg(n)) each
  - n Inserts O(lg(n)) each
  - m-n Decrease-keys O(lg(n)) each
  - m Do nothings O(1) each
- Total time is O(m lg(n))

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# But wait …suppose we could decrease-key in time O(1)

- Heap operations across algorithm:
  - n Extract-mins O(lg(n)) each
  - n Inserts O(lg(n)) each
  - m-n Decrease-keys ~~O(lg(n))~~ O(1) each
  - m Do nothings O(1) each
- Then total time is O(m + n lg(n))

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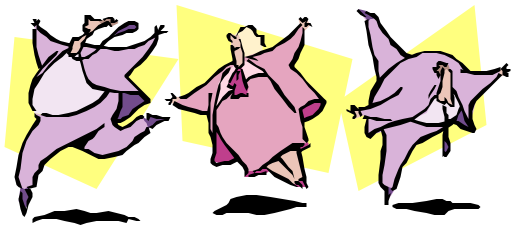
## Bravo, bravo …





## Do It With Fibonacci Heaps

Huh?

Don't worry– it works!



## Kruskal's (Greedy) Algorithm

Let $e_1, e_2, \ldots, e_m$ be the edges of G sorted by increasing weight.

F=V (F is a forest of isolated vertices)

For i=1 to m

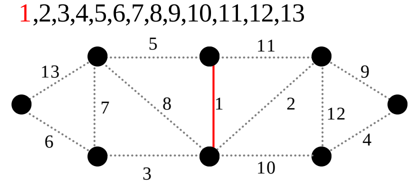
If $F+\{e_i\}$ is acyclic then $F=F+\{e_i\}$.

Return(F)



## Kruskal's Algorithm

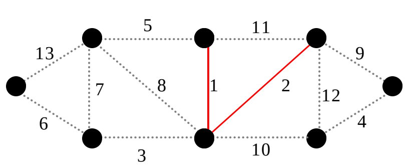
- Order the edge weights. (In this graph the weights are unique.)
- 1,2,3,4,5,6,7,8,9,10,11,12,13






## Kruskal's Algorithm-cont.

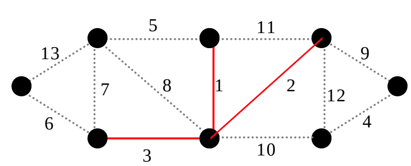
- 1,2,3,4,5,6,7,8,9,10,11,12,13





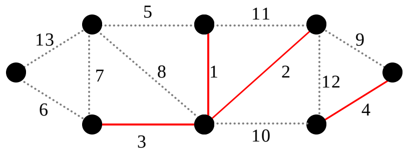

## Kruskal's Algorithm-cont.

- 1,2,3,4,5,6,7,8,9,10,11,12,13



## Kruskal’s Algorithm-cont.

- 1,2,3,4 ,5,6,7,8,9,10,11,12,13

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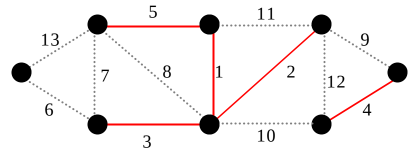
## Kruskal’s Algorithm-cont.

- 1,2,3,4,5,6,7,8,9,10,11,12,13

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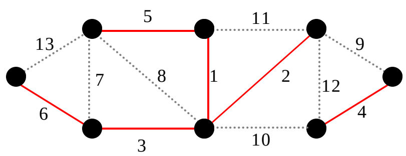
## Kruskal’s Algorithm-cont.

- 1,2,3,4,5,6 ,7,8,9,10,11,12,13

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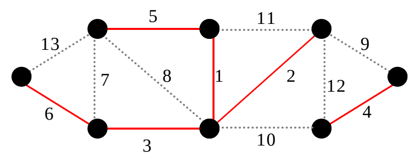
## Kruskal’s Algorithm-cont.

- 1,2,3,4,5,6,7,8,9,10,11,12,13
- Can we add the edge with weight 7?

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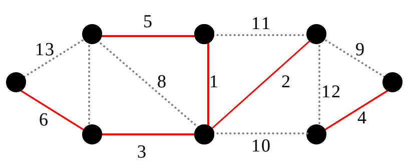
## Kruskal’s Algorithm-cont.

- 1,2,3,4,5,6,7,8,9,10,11,12,13
- Can we add the edge with weight 8?

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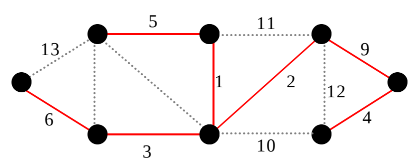
## Kruskal’s Algorithm-cont.

- 1,2,3,4,5,6,7,8,9,10,11,12,13

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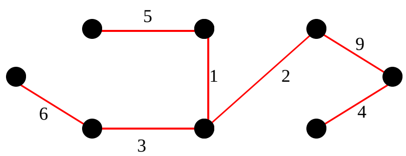
## Kruskal's Algorithm-cont.

MST of G with cost _______






## Kruskal's Algorithm

- Does it work in general?
- Prove it.



## Kruskal's Algorithm
## Proof of Correctness

Claim:

At each stage of the algorithm F is a subgraph of some MST of G.



## Kruskal's Algorithm

Let $e_1, e_2, \ldots, e_m$ be the edges of G sorted by increasing weight.

F=V (F is a forest of isolated vertices)

Claim is true here

For i=1 to m

If F+{$e_i$} is acyclic then F=F+{$e_i$}.

Return(F)



## Loop Invariant

Let $e_1, e_2, \ldots, e_m$ be the edges of G sorted by increasing weight.

F=V (F is a forest of isolated vertices) Claim is true here

If Claim is true here

For i=1 to m

If F+{$e_i$} is acyclic then F=F+{$e_i$}.

Then Claim is true here

Return(F)



## Kruskal's Algorithm
## Proof of Correctness

**Loop Invariant**:

F is a subgraph of some MST of G .

**Proof**:

Consider the $k^{th}$ execution of the loop. Let T be a MST of G containing F. What can happen during the loop?

1. $e_k$ is not added to F
2. $e_k$ is added to F

## Kruskal's Algorithm Proof of Correctness

**Loop Invariant**:

F is a subgraph of some MST of G.

**Proof**:

1. $e_k$ is not added to F

   In this case F does not change so the claim holds when execution of loop concludes



## Kruskal's Algorithm Proof of Correctness

**Loop Invariant**:

F is a subgraph of some MST of G.

**Proof**:

2. $e_k$ is added to F

   We know that T is a MST of G and T contains F.

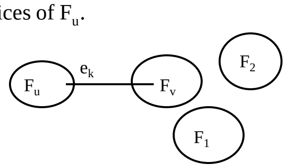
   Need to show there is a MST of G that contains F+{ $e_k$ }

   If $e_k$ is an edge of T we are done. So assume not.



## What do we know?

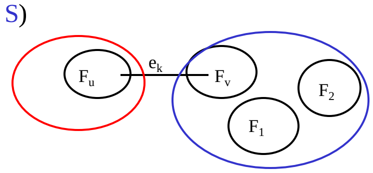
Assume $e_k$=(u,v). The vertices u and v are in separate connected components. Let S be the vertices of $F_u$.






## What do we know?

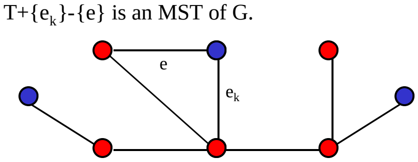
$e_k$ is a minimum weight edge spanning (S,V-S)






## Using our tree facts

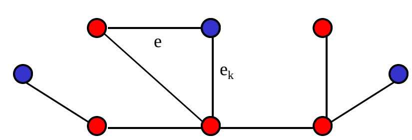
- The graph T+{$e_k$} contains exactly one cycle.
- This cycle contains $e_k$ and at least one additional edge e that spans (S,V-S).
- T+{$e_k$}-{e} is an MST of G.






## Moreover

- T+{$e_k$}-{e} is an MST of G **that contains the edges of F+{$e_k$}.**



## Running Time

- We’ll save that for later…