CS140: Algorithms

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Lecture 12

Algorithm Design Techniques

- Induction (Self-Reduction)
  - Divide and Conquer
  - **Dynamic Programming**

Outline

- **Longest Common Subsequence**
  - Inductive Approach
  - Dynamic Programming
  - Backtracking
- Matrix Chain Multiplication

Longest Common Subsequence

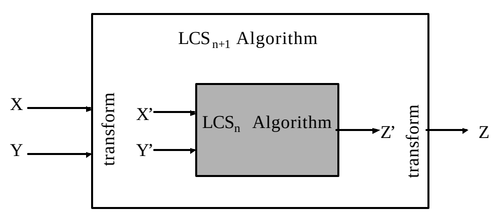
- Input: Two sequences (lists) of integers
  $\mathbf{X}=x_1,x_2, \ldots , x_j$ and $\mathbf{Y}=y_1,y_2, \ldots , y_m$
- Output: A longest subsequence of X that is also a subsequence of Y

LCS - Example

- Input: X = 1, -2, 3, 4, 9, 18
  Y = 3, 9, 1, -2, 5, -2, 22, 18
- Output: Z = 1, -2, 18

LCS_{n+1} Algorithm

Finds LCS of sequences with n+1 or fewer elements (total)



Easy cases:
X or Y is empty

- LCS(Φ, Y[1…m]) =
- LCS(X[1…j], Φ) =

Harder case
(Assume j>0, m>0)

X = $x_1, x_2, \ldots, x_{j-1}, x_j$ **Y** = $y_1, y_2, \ldots, y_{m-1}, y_m$

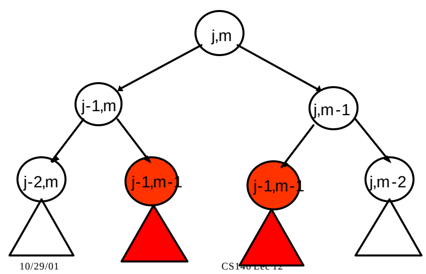
1. $x_i = y_m$:

2. $x_i \neq y_m$:

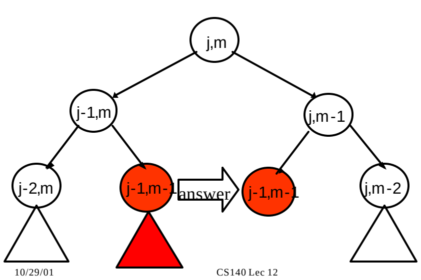
Run Time

T(j,m) =max(T(j-1,m) + T(j,m-1)), T(m-1,n-1)) + c
$\gtrsim$ 2T(j-1,m-1) + c
= $\Omega(2^{\min(j,m)})$

Run Time Analysis
Many duplicated subtrees



Dynamic Programming
Don't Recalculate



Dynamic Programming

- A(i,k) is the length of a longest common subsequence of X[1…i] and Y[1…k]
- A(i,0)=A(0,k)=0 0≤i≤j and 0≤k≤m
- A(i,k)= maximum of
  A(i-1,k), A(i,k-1), and A(j-1,k-1)+match(x_i,y_k)
- A(j,m) is the length of a longest common subsequence of X and Y

A(i,k)

		1	-2	3	4	9	18
	0	0	0	0	0	0	0
3	0	0	0	1	**1**	**1**	1
9	0	0	0	1	**1**		
1	0						
-2	0						
18	0						

A(i,j)

		1	-2	3	4	9	18
	0	0	0	0	0	0	0
3	0	0	0	1	1	1	1
9	0	0	0	1	1	2	2
1 -2	0	1	1	1	1	2	2
5	0	1	2	2	2	2	2
18	0	1	2	2	2	2	3

LCS - Algorithm

LCS($X=x_1,x_2,\ldots,x_j;Y=y_1,y_2,\ldots,y_m$)

```
For i=0 to j: A(i,0)=0
For i=0 to m: A(0,i)=0
For i=1 to j
    For k=1 to m
        If x_i=y_k then match=1 else match=0
        A(i,k) =max(A(i-1,k),A(i,k-1),A(i-1,k-1)+match))
Return A(j,m)
```


Run Time Analysis

- Number of table entries:
- Time to compute one entry:
- Run time:

Backtracking

		1	-2	3	4	9	18
	0	0	0	0	0	0	0
3	0	0	0	1	1	1	1
9	0	0	0	1	1	2	2
1	0	1	1	1	1	2	2
-2	0	1	2	2	2	2	2
18	0	1	2	2	2	2	3

Outline

- Longest Common Subsequence
  - Inductive Approach
  - Dynamic Programming
  - Backtracking
- **Matrix Chain Multiplication**

Matrix Chain Multiplication

- A is an $n \times m$ matrix
- B is an $m \times k$ matrix
- How many scalar multiplications are needed to compute AB?

Matrix Chain Multiplication

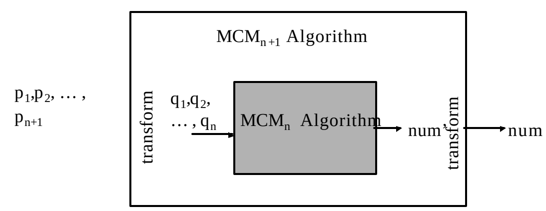
- A is a 2×5 matrix
- B is a 5×1000 matrix
- C is a 1000×2 matrix.
- How many scalar multiplications are needed to compute ABC?
  - (AB)C
  - A(BC)

Matrix Chain Multiplication

- Input: A list of $n+1$ integers $p_1, p_2, \dots, p_{n+1}$
- Output: The minimum number of scalar multiplications needed to compute $\Pi_{i=1\dots n} A_i$ where A_i is a $p_i \times p_{i+1}$ matrix.

(Assume a standard matrix multiplication procedure is used; i.e. no Strassen-like improvements.)

MCM_{n+1} Algorithm



Inductive Approach

- Consider an input: $p_1, p_2, p_3, p_4, p_5, p_6$
- Imagine an optimal solution: 10773

Inductive Approach

- Consider an input: $p_1, p_2, p_3, p_4, p_5, p_6$
- Imagine an optimal way of multiplying matrices A_1, A_2, A_3, A_4, A_5:

$$(A_1(A_2A_3))\,(A_4A_5)$$

Inductive Approach

- There is some last multiplication
  $(A_1(A_2A_3)) \mid (A_4A_5)$

Inductive Approach cont.

- There is some last multiplication
  $(A_1(A_2A_3)) \mid (A_4A_5)$
- So $OPT(A_1,A_2,A_3,A_4,A_5) =$
  $OPT(A_1,A_2,A_3) + OPT(A_4,A_5) + p_1p_4p_5$

Inductive Approach

- We don't know where the top split occurs … but clearly $OPT(A_1,A_2,A_3,A_4,A_5) =$

$$\min_{0<k<5} OPT(A_1,\ldots,A_k) + OPT(A_{k+1},\ldots,A_5) + p_1p_{k+1}p_5$$

- where $OPT(A)=0$

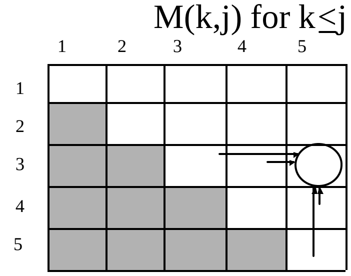
Running Time

- $T(n) = \Sigma_{0<k<n} T(k) + T(n-k) + c$
  $\geq 2T(n-1) + c$
  $= \Omega(2^n)$

Dynamic Programming

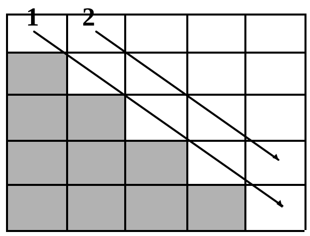
- Use a table to store results
- What kind of results?
  - $M(k,j)$ = Minimum number of multiplications to compute $\Pi_{i=k \ldots j} A_i$

M(k,j) for $k \leq j$



M(3,5) needs:
M(3,3),M(4,5),
M(3,4),M(5,5)

M(k,j) needs M(i,m) where m-i < j-k



M(k,j) needs M(i,m) where m-i < j-k

0				
	0			
		0		
			0	
				0

Dynamic Programming Algorithm

M(k,k)=0

For j,k such that j-k = 1, 2, …, n-1

$M(k,j)= \min_{i=k \dots j-1} M(k,i)+M(i+1,j) + p_k p_{i+1} p_j$

Return M(1,n)

Input: 2,3,1,5,4,8 (A_1 is 2x3, A_2 is 3x1, …)

0	6			
	0	15		
		0	20	
			0	160
				0

MCM Algorithm

- Recursive Algorithm takes exponential time.
- Dynamic Programming takes _________.