NP-completeness
Problem A is NP-Complete if
• A is in NP
• A is NP-hard

A ∈ NP
A is a decision problem and its yes-instances can be described as
\[ \{ x \mid \exists y \text{ such that } P(x,y) \} \]
for a polynomial-time checkable predicate P.

Clique
• Input: A graph G and an integer K
• Question: Does G contain a fully-connected subgraph with K or more vertices?

Clique is in NP
Proof: Show how to express the yes-instances of Clique as \( \{ x \mid \exists y \text{ such that } P(x,y) \} \).
• x: G,K
• y: A subset W of the vertices of G
• P(x,y)=P(G,W):
  – \( |W|\geq K \)
  – Each pair of vertices of W is connected by an edge in G

Independent Set
• Input: A graph G and an integer K
• Question: Does G have K or more vertices that mutually non-adjacent?
Independent is in NP

**Proof:** Show how to express the yes-instances of Independent Set as \( \{ x | \exists y \text{ such that } P(x,y) \} \).
- \( x \): \( G,K \)
- \( y \): A subset \( W \) of the vertices of \( G \)
- \( P(x,y) = P(G,W) \):
  - \( |W| \geq K \)
  - No pair of vertices in \( W \) is connected by an edge in \( G \)

Vertex Cover

- **Input:** A graph \( G \) and an integer \( K \)
- **Question:** Does \( G \) have \( K \) or fewer vertices that touch every edge of \( G \)?

Vertex Cover is in NP

**Proof:** Show how to express the yes-instances of Vertex Cover as \( \{ x | \exists y \text{ such that } P(x,y) \} \).
- \( x \): \( G,K \)
- \( y \): A subset \( W \) of the vertices of \( G \)
- \( P(x,y) = P(G,W) \):
  - \( |W| \geq K \)
  - Every edge of \( G \) is incident to some vertex in \( W \)

NP-completeness

Problem A is NP-Complete if
- A is in NP
- A is NP-hard

A is NP-hard

For any problem \( B \in \text{NP} \), \( B \preceq^p A \).

To prove A is NP-hard

Find some NP-hard problem \( B' \) and show that \( B' \preceq^p A \). Then for any \( B \in \text{NP} \), \( B \preceq^p A \).
Reduction Map

Clique
  ↓
Independent Set
  ↓
Vertex Cover
  ↓
Dominating Set

Legend

VC $\propto_p$ DS:
1. If VC is NP-hard then so is DS.
2. If DS can be solved efficiently then so can VC.

NP-Completeness Map

Clique
  ↓
Independent Set
  ↓
Vertex Cover
  ↓
Dominating Set

Clique $\leftrightarrow$ Independent Set

• A clique in G is an independent set in $G'$.
• A clique in $G'$ is an independent set in G.

Clique $\propto_p$ Independent Set

For $G = (V,E)$ the complement of G is $G' = (V, V \times V - E)$

Clique $\leftrightarrow$ Independent Set

Algorithm for Ind. Set

Algorithm for Clique

G

Y

N

G'$^c$
Independent Set $\preceq_p$ Clique

Algorithm for Independent Set

Algorithm for Clique

Gk

$G',k$

Y

N

Algorithm for Clique

Algorithm for Independent Set

NP-Completeness Map

Clique

trivial

Independent Set

simple

Vertex Cover

easy

Dominating Set

Vertex Cover $\leftrightarrow$ Independent Set

For $G = (V,E)$:

$W \subseteq V$ is a Vertex Cover of $G$ iff $V - W$ is an Independent Set of $G$

Examples:

Every edge of $G$ is incident to a vertex in $W$ iff no edge of $G$ has both endpoints in $V - W$.

$T_{clique}(n,m) = n^2 + T_{ind-set}(n,n^2-m)$

If $T_{ind-set}$ is polynomially-bounded then so is $T_{clique}$.

It $T_{clique}$ is polynomially-bounded then so is $T_{ind-set}$.

Vertex Cover $\leftrightarrow$ Independent Set

Vertex Cover $\preceq_p$ Independent Set

Note: $G$ has $n$ vertices.
Independent Set $\preceq_p$ Vertex Cover

$G,k$ \quad Y \quad N

Algorithm for Independent Set
Algorithm for Clique

Note: $G$ has $n$ vertices.

Reduction Map

Clique
Independent Set
Vertex Cover
Dominating Set

Transform

Add new vertex for each edge

$G, K \rightarrow G', K' = K + I$,
$I$ is the number of isolated vertices of $G$

Claim

- $G$ has a vertex cover of size $K$ or less iff $G'$ has a dominating set of size $K+I$ or less.

Proof $\Rightarrow$

Let $W$ be a vertex cover of $G$ containing $K$ or fewer vertices. Let $W'$ be the isolated vertices of $G$. Claim: $W \cup W'$ is a dominating set of $G'$ containing $K+I$ or fewer vertices.
- Let $v$ be a vertex of $G$. We must show that either $v \in W \cup W'$ or $v$ is adjacent to a vertex in $W \cup W'$.
- If $v$ is an isolated vertex then $v \in W'$ and we are done.
- Suppose $v$ is not an isolated vertex. Then we may assume that $(u,v)$ is an edge of $G$. Since $W$ is a vertex cover of $G$ either $v \in W$ or $u \in W$. In either case we are done.
Proof ⇐
Let W be a dominating set of G' containing K+I or fewer vertices.
Then there exists a dominating set of G' containing K+I or fewer vertices all of which are vertices of G.

Proof ⇐
Let W be a dominating set of G' containing K+I or fewer vertices. WLOG assume W only contains vertices of G.
Let W' be the isolated vertices of G. Then W-W' is a vertex cover of G containing K or fewer vertices.
Let e=(u,v) be an edge of G. We must show that either u or v is in W-W'.
By our reductions, X_{u,v} is a vertex of G'. By our previous argument, X_{u,v} is not in W. Since W is a dominating set of G', either u or v is in W'. Neither u nor v is isolated so either u or v is in W-W'.

Claim
• The reduction can be done in polynomial time.

NP-completeness
Problem A is NP-Complete if
• A is in NP
• A is NP-hard
Show B \leq A, where B is NP-hard.
Where did it all begin?

Cook’s Theorem
• Cook’s Theorem: SAT is NP-hard.

Levin’s Theorem
• Levin’s Theorem: Tiling is NP-hard.
Reduction Map II

Hamiltonian Cycle
↓ ▲ easy
Hamiltonian Path
↓ trivial
Longest Path

Transform

• By definition, G has a Hamiltonian Path iff it has a simple path of length n-1. (n=#vertices)

• So let G’=G and K=n-1

HC \propto_p HP

Add new vertex X attached to some vertex u in G

Transform – Step 1

G

Algorithm for HP

\rightarrow

Algorithm for HC

G’

Y

N

Algorithm for LP

G, K

Algorithm for HP

G

Y

N

Hamiltonian Cycle
↓ ▲ easy
Hamiltonian Path
↓ trivial
Longest Path

Algorithm for  LP

Algorithm for HP
Transform – Step 2

Claim
- G has a Hamiltonian Cycle iff G' has a Hamiltonian Path
- The reduction can be done in polynomial time

Exercise: HP $\propto_p$ HC

Exercise: Reduction Map III