CS140: Algorithms

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Lecture 17

NP-completeness

Problem A is NP-Complete if
- A is in NP
- A is NP-hard

A ε NP

A is a decision problem and its yes-instances can be described as

$\{x \mid \exists y \text{ such that } P(x,y)\}$

for a polynomial-time checkable predicate P.

Clique

- Input: A graph G and an integer K
- Question: Does G contain a fully-connected subgraph with K or more vertices?

Clique is in NP

Proof: Show how to express the yes-instances of Clique as $\{x \mid \exists y \text{ such that } P(x,y)\}$.
- x: G,K
- y: A subset W of the vertices of G
- P(x,y)=P(G,W):
  - $\|W\| \geq K$
  - Each pair of vertices of W is connected by an edge in G

Independent Set

- Input: A graph G and an integer K
- Question: Does G have K or more vertices that mutually non-adjacent?

Independent is in NP

Proof: Show how to express the yes-instances of Independent Set as {x | ∃y such that P(x,y)}.

- x: G,K
- y: A subset W of the vertices of G
- P(x,y)=P(G,W):
  - ||W||≥K
  - No pair of vertices in W is connected by an edge in G

Vertex Cover

- Input: A graph G and an integer K
- Question: Does G have K or fewer vertices that touch every edge of G?

Vertex Cover is in NP

Proof: Show how to express the yes-instances of Vertex Cover as {x | ∃y such that P(x,y)}.

- x: G,K
- y: A subset W of the vertices of G
- P(x,y)=P(G,W):
  - ||W||≥K
  - Every edge of G is incident to some vertex in W

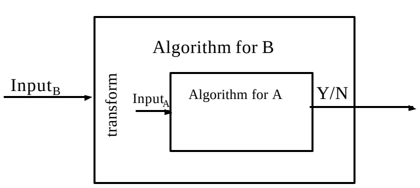
NP-completeness

Problem A is NP-Complete if

- A is in NP
- A is NP-hard

A is NP-hard

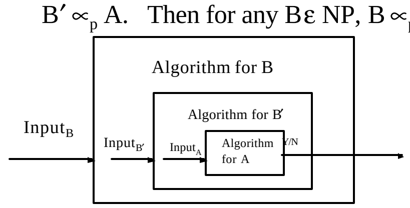
For any problem B ε NP, $B \propto_p A$.



Note: If A can be solved in polynomial time, then so can every other problem in NP.

To prove A is NP-hard

Find some NP-hard problem B′ and show that $B' \propto_p A$. Then for any B ε NP, $B \propto_p A$.



Reduction Map

Clique
↕
Independent Set
↕
Vertex Cover
↓
Dominating Set

Reduction Map Legend

Vertex Cover
↓
Dominating Set

VC $\propto_p$ DS:

1. If VC is NP-hard then so is DS.
2. If DS can be solved efficiently then so can VC.

NP-Completeness Map

Clique
↕ trivial
Independent Set
↕ simple
Vertex Cover
↓ easy
Dominating Set

Clique ↔ Independent Set

For G = (V,E) the complement of G is

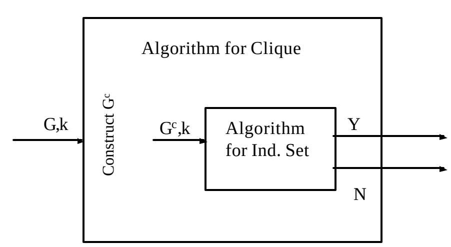
$G^c = (V, V \times V - E)$

G G^c

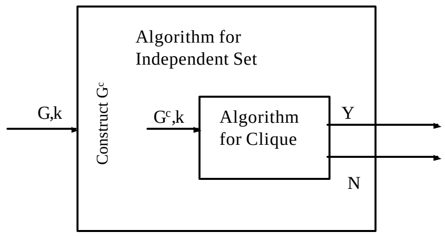
Clique ↔ Independent Set

- A clique in G is an independent set in G^c.
- A clique in G^c is an independent set in G.

Clique $\propto_p$ Independent Set



Independent Set $\propto_p$ Clique

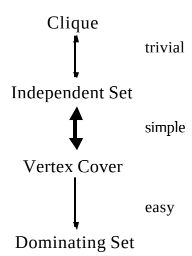


T_{clique} (n,m)

T_{clique} (n,m)= $n^2 + T_{ind\text{-}set}(n,n^2\text{-}m)$

If $T_{ind\text{-}set}$ is polynomially-bounded then so is T_{clique}.

NP-Completeness Map

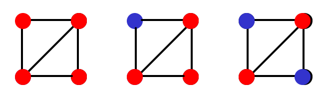


Vertex Cover ↔ Independent Set

For G = (V,E):

W⊆V is a Vertex Cover of G iff V-W is an Independent Set of G

Examples:

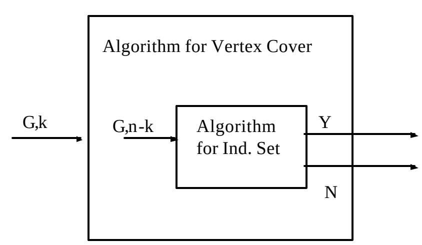


Vertex Cover ↔ Independent Set

W⊆V is a Vertex Cover of G iff V-W is an Independent Set of G.

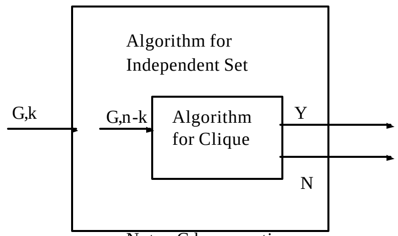
Every edge of G is incident to a vertex in W iff no edge of G has both endpoints in V-W.

Vertex Cover $\propto_p$ Independent Set



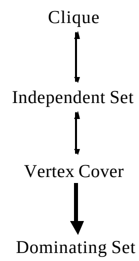
Note: G has n vertices.

Independent Set $\propto_p$ Vertex Cover

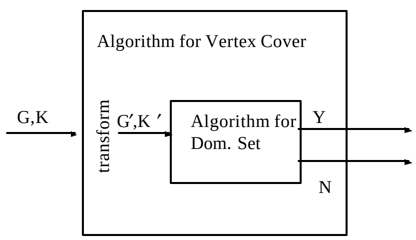


Note: G has n vertices.

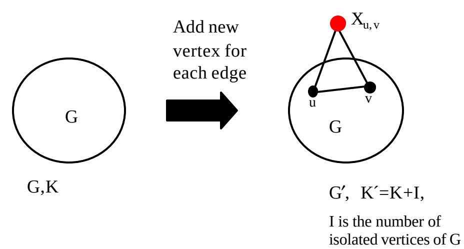
Reduction Map



Vertex Cover $\propto_p$ Dominating Set



Transform



Claim

- G has a vertex cover of size K or less iff G′ has a dominating set of size K+I or less.

Proof $\Rightarrow$

Let W be a vertex cover of G containing K or fewer vertices. Let W′ be the isolated vertices of G. Claim: $W \cup W'$ is a dominating set of G′ containing K+I or fewer vertices.

- Let v be a vertex of G′. We must show that either $v \in W \cup W'$ or v is adjacent to a vertex in $W \cup W'$.
- If v is an isolated vertex then $v \in W'$ and we are done.
- Suppose v is not an isolated vertex. Then we may assume that (u,v) is an edge of G. Since W is a vertex cover of G either $v \in W$ or $u \in W$. In either case we are done.

Proof ⇐

Let W be a dominating set of G′ containing K+I or fewer vertices.

Then there exists a dominating set of G′ containing K+I or fewer vertices all of which are vertices of G.

Proof ⇐

Let W be a dominating set of G′ containing K+I or fewer vertices. WLOG assume W only contains vertices of G.

Let W′ be the isolated vertices of G. Then W-W′ is a vertex cover of G containing K or fewer vertices

Let e=(u,v) be an edge of G. We must show that either u or v is in W-W′.

By our reductions, $X_{u,v}$ is a vertex of G′. By our previous argument, $X_{u,v}$ is not in W. Since W is a dominating set of G′, either u or v is in W. Neither u nor v is isolated so either u or v is in W-W′.

Claim

- The reduction can be done in polynomial time.

NP-completeness

Problem A is NP-Complete if

- A is in NP
- A is NP-hard

Show $B \propto_p A$, where B is NP-hard.

Where did it all begin?

Cook’s Theorem

- Cook’s Theorem: SAT is NP-hard.

- SAT:
  - Input: A boolean expression E in conjuctive normal form.
  - Question: Does E have a satisfying truth assignment; i.e. a truth assignment under which E evaluates to true?

Levin’s Theorem

- Levin’s Theorem: Tiling is NP-hard.

Reduction Map II

Hamiltonian Cycle

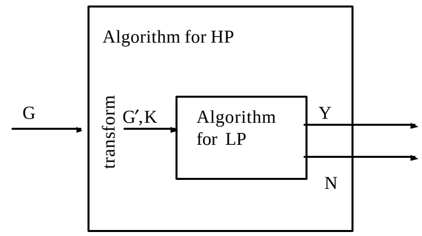
↓ ↑ easy

Hamiltonian Path

↓ trivial

Longest Path

HP $\propto_p$ LP



Transform

- By definition, G has a Hamiltonian Path iff it has a simple path of length n-1. (n=#vertices)
- So let G'=G and K=n-1

Reduction Map II

Hamiltonian Cycle

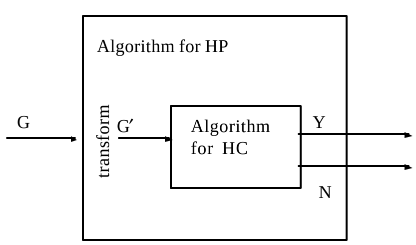
↓ ↑ easy

Hamiltonian Path

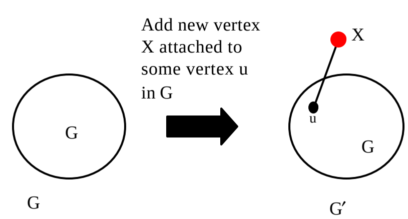
↓ trivial

Longest Path

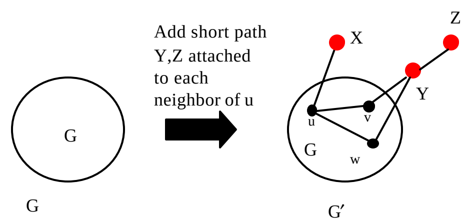
HC $\propto_p$ HP



Transform – Step 1



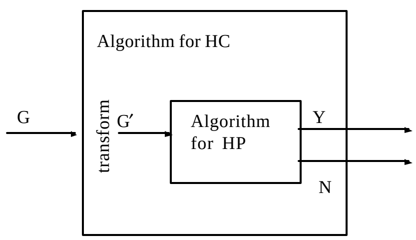
Transform – Step 2



Claim

- G has a Hamiltonian Cycle iff G’ has a Hamiltonian Path
- The reduction can be done in polynomial time

Exercise: HP $\propto_p$ HC



Exercise: Reduction Map III