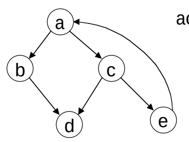
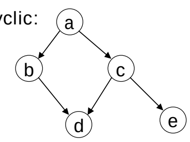
Another Function Decomposition Example (which might use anonymous functions)

- Construct a function that will tell whether a directed graph, represented as a list of arcs, is acyclic.

cyclic:



acyclic:

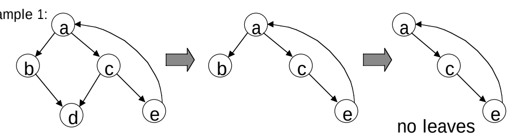


"Pruning" Method

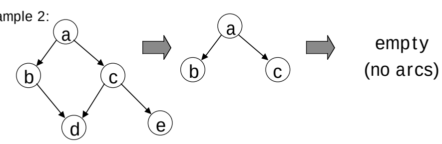
- Rosalind B. Marimont
  A new method of checking the consistency of precedence matrices
  Journal of the ACM **6**, 164-171, 1959
- Pruning away any arcs that point to leaves does not change the cyclic/acyclic nature of the graph.
- Pruning such arcs may produce additional leaves.
- Prune until no further pruning is possible:
  - If the result is empty, the original graph was acyclic.
  - If not, it was cyclic.

Examples of Pruning (leaves shown in green)

example 1:



example 2:

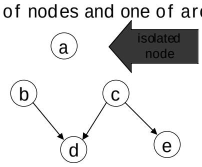


Pruning with Graphs as Lists

- Example 1:
  - [[a, b], [a, c], [b, d], [c, d], [c, e], [e, a]] ➡
  - [[a, b], [a, c], [c, e], [e, a]] ➡
  - [[a, c], [c, e], [e, a]] (no leaves)
- Example 2:
  - [[a, b], [a, c], [b, d], [c, d], [c, e]] ➡
  - [[a, b], [a, c]] ➡
  - [ ]

Note

- We are assuming that every node in the graph is on one or the other end of an arc, i.e. there are no isolated nodes, as in the graph below.
- Otherwise, we'd have to represent the graph with two lists: one of nodes and one of arcs.



Functional Code

- Basic idea:
  - As long as there is a leaf:
    Remove leaves and their attached arcs
- Translation:
- isAcyclic(Graph) =
  null(iterate(removeLeaves, hasLeaf, Graph));

  null: test for empty list
  iterate: iterate first arg. as long as second arg. true
  removeLeaves: remove all leaves and their arcs
  hasLeaf: test whether there is a leaf
  Graph: starting graph

hasLeaf

- A Graph has a leaf iff isLeaf is true for one of its nodes.
- hasLeaf(Graph) =
  some((Node)=>isLeaf(Node, Graph), nodes(Graph));

test whether first arg. is true for some element of second arg.

true when Node is a leaf of this Graph

list of nodes of Graph

isLeaf

- A node is a leaf if it is not the first of any arc in the graph.
- isLeaf(Node, Graph) =
  !member(Node, map(first, Graph));

'not' operator

all nodes of Graph that begin some arc

nodes(Graph)

- nodes(Graph) =
  remove_duplicates(append(map(first, Graph),
  map(second, Graph)));
- remembering our assumption: that every node in the graph is on one or the other end of an arc, i.e. there are no isolated nodes, as in the graph below.

remove_leaves

- To remove the leaves:
  remove any arc that points to a leaf
- removeLeaves(Graph) =
  drop((Arc)=>isLeaf(second(Arc), Graph),
  Graph);

the node to which Arc points

the list of arcs in the graph

iterate

- iterate(action, continue, State) =
  continue(State) ?
  iterate(action, continue, action(State))
  : State;

conditional expression (as in C++, Java)
P ? A : B
means if P is true then the value of the expression is A; otherwise it is B.

Low-Level
Functional Programming

What's "Low-Level" About This?

- "low-level" refers to the construction of functions by explicitly composing and decomposing lists.
- Previously we used higher-order functions to do most of the non-trivial work in a functional decomposition.
- Now we are going to use pattern matching rules, recursion, etc.

Fundamental List Dichotomy

- A list is either:
  - **empty**, [ ]   or
  - **non-empty**, in which case it has both a
    - first
    - rest
- Most list definitions deal with these cases separately.
- Definitions are typically a form of **inductive definition**, in which [ ] is the basis.

List Decomposition Notation

- When a list is non-empty, it has a first element and the rest of the elements form a list.
- The general ***form*** of a **non-empty list** will be represented:

  [F | R]

  Here **F** is a variable represents the first element, and **R** is a variable representing the rest of the elements (**R** has a list as its value, even though brackets aren't around R).

List Decomposition Example

- Consider a defining equation:

  [F | R] = [1, 2, 3, 4]

  F is a variable represents the first element, so:

  F == 1

  R is a variable representing the rest of the elements, so:

  R == [2, 3, 4]

List Decomposition Clarified

- A defining equation:

  [F | R] = *some list*

  can only be valid when the RHS list is non-empty.

  Thus

  [F | R] = [ ] can *never* be a valid equation.

Defining Functions by Rules

- Suppose we want to define a function taking an arbitrary list as an argument.
- It is sufficient to:
  - define the function on the empty list, and

    called the "basis"
  - define the function on a general non-empty list.

    called the "induction step" or "recursion"

Example

- Define the function **halve_all,** which divides every element in a list by 2.
  - halve_all([ ]) => [ ];
  - halve_all([F | R]) => [F/2 | halve_all(R)];
- This can be read:
  - "halving all of the empty list is the empty list."
  - "halving all of a non-empty list is half of the first element ***followed by*** halving all of the rest."

Computation by "Rewriting"

- halve_all([2, 4, 6]) ➡
- [1 | halve_all([4, 6])] ➡
- [1 | [2 | halve_all([6])]] ➡
- [1 | [2 | [3 | halve_all([ ])]]] ➡
- [1 | [2 | [3 | [ ]]]] ==
- [1 | [2 | [3]]] ==
- [1 | [2, 3]] ==
- [1, 2, 3]

Extended Notation for Greater Readability

- The first so-many, rather than just the first, element, can be shown separated by commas:
  - [a, b, c, d | R] means a list with at least 4 elements, a, b, c, d, followed by the elements in list R (which could be empty).
- In the extended notation:
  - halve_all([2, 4, 6]) ➡
  - [1 | halve_all([4, 6])] ➡
  - [1, 2 | halve_all([6])] ➡
  - [1, 2, 3 | halve_all([ ])] ➡
  - [1, 2, 3]

A Way of Remembering

- The combination

  | [...]

  inside a list "melts away" into

  , ...

  unless ... is empty, then it just melts away
- Examples:
  - [1 | [2, 3, 4]] == [1, 2, 3, 4]
  - [1, 2 | [3 , 4]] == [1, 2, 3, 4]
  - [1, 2, 3 | [4]] == [1, 2, 3, 4]
  - [1, 2, 3, 4 | [ ]] == [1, 2, 3, 4]

Alternate

- Of course, we could have just used ***map*** *in this particular case*:
  - halve(A) = A/2;
  - halve_all(X) = map(halve, X);
- Use higher order functions such as *map* when possible; resort to lower-order ones when you think you need to.
- Higher-order functions can often tell the story more succinctly.

Example

- Define the function **member** which tests whether the first argument is an element of the list in the second argument.
  - member(X, [ ]) => 0;
  - member(X, [F | R]) =>

    $\underbrace{\text{(X == F) ? 1 : member(X, R);}}$

    conditional expression (as in C++, Java)

Alternate

- Instead of using a conditional expression, use a third rule with **pattern matching**:
  - member(X, [ ]) => 0;
  - member(X, [X| R]) => 1; Note: X's must match
  - member(X, [F| R]) => member(X, R);
- The rule used is always the **first** (from top to bottom) applicable one.

Rule Matching

- Consider evaluating
  - member(3, [1, 2, 3, 4]) ➡ rule 3 is the first to apply
  - member(3, [2, 3, 4]) ➡ rule 3 is the first to apply
  - member(3, [3, 4]) ➡ rule 2 is the first to apply
  - 1

```
member(X, [ ]) => 0;                  // rule 1
member(X, [X| R]) => 1;               // rule 2
member(X, [F| R]) => member(X, R);    // rule 3
```

Rule Matching

- Consider evaluating
  - member(5, [1, 2, 3]) ➡ rule 3 is the first to apply
  - member(5, [2, 3]) ➡ rule 3 is the first to apply
  - member(5, [3]) ➡ rule 3 is the first to apply
  - member(5, []) ➡ rule 1 is the first to apply
  - 0

Second Alternate
(less desirable)

- Use a **conditional guard**:
  - member(X, [ ]) => 0;
  - member(X, [F| R]) => $\underbrace{\text{(X == F)}}_{\text{conditional guard}}$? 1;
  - member(X, [F| R]) => member(X, R);
- The condition is tested after any other matching is applied.
- If the condition fails, then subsequent rules are tried.

Matching with Two or More List Arguments

- Some functions have more than one list argument.
- Induction might, or might not, use rules that dichotomize both lists.

Example: List Equalty First Rule

- Two lists are equal if they both are empty:

  equals([ ], [ ]) => 1;

List Equalty: Second Rule

- Two lists are equal if they are both non-empty and
  - the first elements of each are the same, and
  - the lists of the rest of the elements of each are equal.

equals([A | L], [A | M]) => equals(L, M);

List Equalty: Third Rule

- Otherwise, the two lists are not equal:

  equals(X, Y) => 0;

Summary of Equality Rules

1. equals([ ], [ ]) => 1;
2. equals([A | L], [A | M]) => equals(L, M);
3. equals(X, Y) => 0;

Example of List Equality

- Revisit our earlier example:
  - Are these lists equal:

    [1, 2, 3] vs. [1, 2] ?
- Try the rules:
  - equals([1, 2, 3], [1, 2]) ➨ (rule 2)
  - equals([2, 3], [2]) ➨ (rule 2)
  - equals([3], [ ]) ➨ (rule 3)
  - 0
- i.e. the lists are not equal.

Example: Low-Level Version of Nim *magic*

- Recall:

  magic(L, s) = a new list similar to L, except that the ***first*** element, p, where xor(p, s) < p, is ***replaced*** with xor(p, s).
- Translation to rules:
  - magic([ ], s) => [ ];
  - magic([p | R], s) => xor(p, s) < p ? [xor(p, s) | R];
  - magic([p | R], s) => [p | magic(R, s)];

Improving Efficiency

- Recall the ***isAcyclic*** example.
- There there might be occasion to compute the same thing multiple times, for example

  isLeaf(Node, Graph)

  may be called **multiple times** for a given Graph:
  - hasLeaf(Graph) = some((Node)=>isLeaf(Node, Graph), nodes(Graph));
- **Each** time isLeaf is called, map(first, Graph) is recomputed
  - isLeaf(Node, Graph) = !member(Node, map(first, Graph));
- It may be better to compute map(first, Graph) “up front” and pass it to isLeaf.

Improving Efficiency

- Computing up front means an extra argument to isLeaf, which will may clutter the meaning of a given function:
- Below we "promote" map(first, Graph) out of isLeaf
  - hasLeaf(Graph) =
    some((Node)=>isLeaf(Node, map(first, Graph)), nodes(Graph));
  - isLeaf(Node, Firsts) = !member(Node, Firsts);
- However, it is still may be called once for each node.
- In order to avoid recomputation, we need to promote it out of the call to *some*.
- This can be done with a **local equation**, or "equational guard":
  - hasLeaf(Graph) =
    Firsts =map(first, Graph),
    some((Node)=>isLeaf(Node, Firsts), nodes(Graph));

equational guard

Improving Efficiency

- If we prefer not to use an equational guard, we can pass Firsts as an argument to hasLeaf:
- Below we "promote" map(first, Graph) out of isLeaf
  - hasLeaf2(Graph, Firsts) =
    some((Node)=>isLeaf(Node, Firsts), nodes(Graph));
- This will necessitate introduction of a new definition for the original 1-argument hasLeaf:
  - hasLeaf(Graph) = hasLeaf2(Graph, map(first, Graph));

Improving Efficiency

- Alas, we overlooked at least one little detail:
- *isLeaf* is used in removeLeaves as well as in hasLeaf, so we'll similarly have to adjust its use there.
  - removeLeaves(Graph) =
    removeLeaves2(Graph, map(first, Graph));
  - removeLeaves2(Graph, Firsts) =
    drop(((Arc)=>isLeaf(second(Arc), Firsts)), Graph);

Improving Efficiency

- There is still one obvious inefficiency:
  - map(first, Graph) is computed in both hasLeaf and removeLeaves; We'd prefer to compute it only once.
  - The low-level definition (using recursion instead of *iterate*) then might be:
    - isAcyclic(Graph) =
      Firsts = map(first, Graph),
      hasLeaf2(Graph, Firsts) ?
      isAcyclic(removeLeaves2(Graph, Firsts))
      : null(Graph);
  - Alternatively, we could construct a different version of iterate, but it would seem to be rather special purpose.

Closing Notes on Efficiency

- In previous slides, we used the property of ***referential transparency*** of functional languages (that expressions can be substituted for equivalent expressions) to improve efficiency.
- It would not generally be possible to do such substitutions in an imperative language; procedures that have side-effects cannot be substituted so freely.
- Transforming a program in the manner shown may impair its clarity and readability; so it is better to maintain a perhaps less-efficient reference version apart from the "production" version.

Mixed Functional Programming Examples

- Use low-level or high-level, whatever fits best
  - Maybe start with low-level, and the use high-level retrospectively
- Radix conversion
  - Tail recursion
- Tree searching

Convert Number to Binary

- Example:
  - toBinary(37) ➪ [1, 0, 0, 1, 0, 1]
    32 + 0*16 + 0*8 + 4 + 0*2 + 1
- First try: use method discussed in class earlier:
  - divide by 2, record remainder, continue with quotient
  - until 0

Convert Number to Binary

- Rules:
  - toBinary1(0) => [ ];
  - toBinary1(N) => [$\underbrace{\text{N\%2}}_{\text{remainder}}$ | toBinar$\underbrace{\text{y1(N/2)}}_{\text{quotient}}$];
- Problems with this definition?

Convert Number to Binary

- Another try:
  - toBinary(N) = toBinary2(N, []);
  - toBinary2(0, Acc) => Acc;
  - toBinary2(N, Acc) =>
    toBinary2($\underbrace{\text{N/2}}_{\text{quotient}}$, [$\underbrace{\text{N\%2}}_{\text{remainder}}$ | Acc]);
- Why is this definition better?
- What is still lacking?

Accumulators and Tail Recursion

- From previous slide:
  - toBinary2(0, Acc) => Acc;
  - toBinary2(N, Acc) =>
    toBinary2(N/2, [N%2 | Acc]);
- Acc is called an "accumulator" argument:
  - It "accumulates" the result until the basis case is reached, the "unloads" it.
- This type of recursion is called "tail recursion":
  - There is no "cleanup" to be done after the recursive call to toBinary2, and therefore no need to **"stack"** calls.
  - We can effectively "turn over control" to the subordinate call; giving a form of **iteration**.

Accumulators and Tail Recursion

- toBinary2(37, []) ➡
- toBinary2(18, [1]) ➡
- toBinary2(9, [0, 1]) ➡
- toBinary2(4, [1, 0, 1]) ➡
- toBinary2(2, [0, 1, 0, 1]) ➡
- toBinary2(1, [0, 0, 1, 0, 1]) ➡
- toBinary2(0, [1, 0, 0, 1, 0, 1]) ➡
- [1, 0, 0, 1, 0, 1]

- toBinary1(37) ➡
- [1 | toBinary1(18)] ➡
- [1, 0 | toBinary1(9)] ➡
- [1, 0, 1 | toBinary1(4)] ➡
- [1, 0, 1, 0 | toBinary1(2)] ➡
- [1, 0, 1, 0, 0 | toBinary1(1)] ➡
- [1, 0, 1, 0, 0, 1| toBinary1(0)] ➡
- [1, 0, 1, 0, 0, 1| [ ]] ➪
- [1, 0, 1, 0, 0, 1]

Notes:

- Can similarly convert to any given base.
- Can pass the base as an argument.
- Can convert back (from numeral list to number).

Exercise

- Construct fromBinary, e.g.
  - fromBinary([1, 0, 0, 1, 0, 1]) ➪ 37
- Considerations:
  - Do we need an accumulator?
  - Can it be done with tail-recursion?
  - Try it and see.

An Approach

- Write **iterative** pseudo-code, then construct recursive equivalent.
- L = ... list to be converted ...;

```
Result = 0;
while(L != [ ])
    {
    Result = 2*Result + first(L);
    L = rest(L);
    }
... answer is in Result ...
```

- Defining fromBinary3(L, Result):
- fromBinary3([], Result) => Result;
- fromBinary3([F | R], Result) => fromBinary3(R, 2*Result+F);
- fromBinary(L) = fromBinary3(L, 0);

- fromBinary3([1, 0, 0, 1, 0, 1], 0) ➡
- fromBinary3([0, 0, 1, 0, 1], 1) ➡
- fromBinary3([0, 1, 0, 1], 2) ➡
- fromBinary3([1, 0, 1], 9) ➡
- fromBinary3([0, 1], 9) ➡
- fromBinary3([1], 18) ➡
- fromBinary3([], 37) ➡
- 37

Exercise

- What if the list were least-significant bit first?
  - Can you do construct the function?
  - Can you construct a tail-recursive implementation?
- Construct the xor function used in the nim move function.
  - This could entail the essence of toBinary and fromBinary in one definition.

Exercises

- Compare "obvious" and tail-recursive forms of:
  - factorial function (fac(n) = 1*2*3*...*n)
  - length function
  - sum of a list
  - reduce
  - reverse

Essential Non-Tail Recursions

- Some functions don't admit a tail-recursive version (unless *reverse* is used before or after):
  - Examples:
    - map, keep, drop
    - append

append Elimination

- When maximum efficiency is desired, uses of append should be avoided.
- It is often possible to get rid of append by defining versions of functions with an extra accumulator argument.
- Example:

```
nodes(Graph) =
    remove_duplicates(append(map(first, Graph),
                             map(second, Graph)));
```

- Show how to avoid append by generalizing map to take an accumulator.