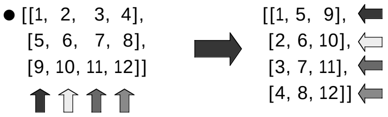
Transposing a Matrix

- By a “matrix” here we mean a list of lists, where:
  - Each inner list has the same length.
- So we are restricting to a 2-D matrix.
- Example:
  - [[1, 2, 3, 4],
    [5, 6, 7, 8],
    [9, 10, 11, 12]]
- Note: A matrix, even in higher dimensions, is a special case of an **unlabeled ordered tree.**

Transposing a Matrix

- By **transposing** a matrix, we mean “flipping” it so that the rows become columns, and vice-versa:
- The transpose of the previous example:



- [[1, 2, 3, 4],
  [5, 6, 7, 8],
  [9, 10, 11, 12]]

➡

[[1, 5, 9],
[2, 6, 10],
[3, 7, 11],
[4, 8, 12]]

How to Transpose Functionally

- The firsts of each row become the first row of the transpose.
- transpose(M) = [map(first, M) | ...???...]
- Now try to get recursion to do the work for us.
- M without the first column is map(rest, M).
- Transposing that gives us the rest of the rows.

How to Transpose Functionally

- transpose(M) =>
  [map(first, M) | transpose(map(rest, M))];
- But we still need a **basis** for the recursion.
- How to represent the “empty” matrix?
- It has no rows or columns, but we need it to be a list of lists.
- It will have this form: [$\overbrace{[], [], \ldots, []}^{\text{all empty}}$]?
- transpose([[ ] | _]) => [ ];

Rule Summary

- transpose([[ ] | _]) => [ ];
- transpose(M) =>
  [map(first, M) | transpose(map(rest, M))];

- For completeness, we may wish to add as a first rule:
- transpose([ ]) => [ ];

Check by Rewriting

- transpose([[1, 2, 3, 4], [5, 6, 7, 8], [9, 10, 11, 12]]) ➡
- [map(first, ...) | transpose(map(rest, ...))] ➡
- [[1, 5, 9] | transpose([[2, 3, 4], [6, 7, 8], [10, 11, 12]])] ➡
- [[1, 5, 9] | [[2, 6, 10] | transpose([[3, 4], [7, 8], [11, 12]])]] ➡
- [[1, 5, 9] | [[2, 6, 10] | [[3, 7, 11] | transpose([[4], [8], [12]])]]] ➡
- [[1, 5, 9] | [[2, 6, 10] | [[3, 7, 11] | [[4, 8, 12] | transpose([[], [], []])]]]] ➡
- [[1, 5, 9] | [[2, 6, 10] | [[3, 7, 11] | [[4, 8, 12] | [ ]]]]] ==
- [[1, 5, 9], [2, 6, 10], [3, 7, 11], [4, 8, 12]] by “melt away” reasoning

- The argument to transpose must have at least one row, even if that row is empty.

Functions Returning Functions

Anonymous Functions Again

- (X) => 5*X
  is a perfectly valid, although anonymous, function
- Functions in rex are “first-class citizens”:
  - They can be passed as **arguments**.
  - They can be put into **lists**.
  - They can be **returned** as values.
- A function-value returning function:
  - scaleBy(F) = (X) => F*X;
  - Here scaleBy(F) returns a function, the function that scales its argument by F.

Use of scaleBy

  - scaleBy(F) = (X) => F*X;
- map(scaleBy(5), [1, 2, 3]) ➪ [5, 10, 15]
- So we could define
  - scale(F, L) = map(scaleBy(F), L);

---

mapster

- mapster(F) = (L) => map(F, L);
- So mapster is a function such that, for any argument function F, returns a function that will map F over a list.

Curried Functions (yum!)

- scaleBy and mapster are examples of “Curried” functions, functions that take their arguments in sequential “tiers”, each returning a function until the last argument.
- Another way to write them in rex is:
  - scaleBy(F)(X) = F*X;
  - mapster(F)(L) = map(F, L);
- This emphasizes that scaleBy(F) has a meaning without the (X).

Anonymous returning Anonymous

- Yet another way to write scaleBy and mapster in rex is:
  - scaleBy = (F) => (X) => F*X;
  - mapster = (F) => (L) => map(F, L) ;
- This clearly shows that the names scaleBy and mapster are incidental; the right-hand sides of the equations can be used as values as they are.

Curried Functions in Math, Science, and Engineering

- In a math, science, or engineering text, you might see

  $f_k(x)$
- Typical nomenclature is that x is the “argument” and k is a “parameter”.
- Actually **both** k and x may be viewed as arguments to f. Typically k is “more fixed”.
- We could define this as f(k)(x) = ...
- Then we can use f(k) as f_k would be used.

A function that “Curries” another function

- Suppose that f represents a typical two-argument function, i.e. f(x, y) is meaningful.
- Define

  curry(f)(X)(Y) = f(X, Y)
- curry(f) is a curried version of f.
- curry(f)(X) is like “f(X, .)” in engineering books.
- Example: scale(K, L) = map(curry(*)(K), L);

  the 2-argument multiply function

A Cool Example

- We know that an **association list** can be viewed as an implementation of a function (with a finite domain):

  [[“Jan”, 31], [“Feb”, 28], [“Mar”, 31], [“Apr”, 30]]
- But we cannot simply **apply** this list as a function:

  [[“Jan”, 31], [“Feb”, 28], [“Mar”, 31], [“Apr”, 30]](“Feb”) ➪ 28

  not in rex, at least.
- We have to use the assoc function instead:

  assoc(“Feb”, ... list above ...) ➪ [“Feb”, 28]

Making Fun of Association Lists

- makeFun(Alist) returns a function that can be applied as an ordinary function:

  f = makeFun([[“Jan”, 31], [“Feb”, 28], [“Mar”, 31], [“Apr”, 30]]);

  f(“Feb”) ➪ 28
- To define makeFun, we use the idea of a function returning a function:

  makeFun(Alist) = (Arg) => second(assoc(Arg, Alist));
- Using makeFun “captures” the Alist in the resulting function.

Composing Functions

- The composition of two functions,

  $f: T \to U$

  $g: S \to T$

  is the function, call it h for now, such that

  $h: S \to U$

  and for every x in S, h(x) = f(g(x))
- The composition is sometimes written using an operator o:

  f o g

Composing Functions

- If anonymous functions are supported, we don’t need the “call it h” aspect:

  compose(f, g)(x) = f(g(x))

  or, equivalently,

  compose(f, g) = (x) => f(g(x))

Composing can make things more efficient in some cases

- map(f, map(g, L)) =

Searching Graphs and Trees

Searching a Graph

- Suppose we want to find whether there is a node with a certain property P reachable from a node in a graph.
- Rather than assume a specific representation such as an arc-list, use **abstraction**:
- Assume we have a function **targets** such that
  targets(Graph, Node)
  is the list of targets of the node.

Finding a node in a graph

- Let findNode be the function to be computed. (call findNode(Graph, Node, P), where Node is the starting node, P is a predicate on nodes.)

- Success indicator convention:
  - If a node is found, return a list of just that node, [Node].
  - If no node is found, return the empty list, [ ].

Finding a node using recursion

- findNode(Graph, Node, P) => P(Node) ? [Node];

- findNode(Graph, Node, P) =>

  findInTarget(targets(Graph, Node), Graph, P);

  i.e. find one among the targets of Node

Finding a node among targets

- findInTargets([ ], Graph, P) => [ ];

- findInTargets([Target | Targets], Graph, P) =>

```
Found = findNode(Graph, Target, P),

Found != [ ] ? Found
             : findInTargets(Targets, Graph, P);
```

local definition

returned value

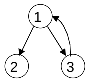
recurse

Mutual Recursion

- The relationship between *find* and *findInTargets* is that of **mutual recursion**.
- *find* delegates work to *findInTargets*
- *findInTargets* delegates work to *find*
- This approach seems natural in this problem.

Correctness

- The previous solution will work if the graph is acyclic.
- If not acyclic, it may work in some cases, and loop infinitely in others.
- So it doesn't *really* "work".
- How can we fix this?



Handling the Cyclic Case

- Since the graph is **finite**, infinite looping can only occur when the **same node** recurs on a path.
- By **keeping track of the nodes on the path from the starting point**, we can check whether a node recurs.

---

find revised

-
  ```
  findNode(Graph, Node, P) = findNode(Graph, Node, P, [ ]);
  ```

-
  ```
  findNode(Graph, Node, P, Seen) =>
          P(Node) ? [Node];
  ```

-
  ```
  findNode(Graph, Node, P, Seen) =>
          member(Node, Seen) ? [ ];
  ```

-
  ```
  findNode(Graph, Node, P, Seen) =>
          findInTargets(targets(Graph, Node), Graph, P, [Node | Seen]);
  ```

---

findInTargets revised

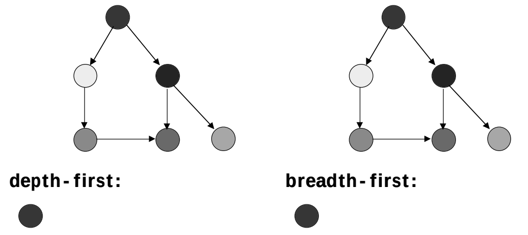
-
  ```
  findInTargets([ ], Graph, P, Seen) => [ ];
  ```

-
  ```
  findInTargets([Target | Targets], Graph, P, Seen) =>

        Found = findNode(Graph, Target, P, Seen),

        Found != [ ] ? Found

                     : findInTargets(Targets, Graph, P, [Target | Seen]);
  ```
- We include Target in the list above so that we never do a recursive find from a node more than once.
- However, we may still search from the same node more than once. Why? Is this preventable?

Varieties of Search

- The version of find presented previously is called **depth-first** search.
- The other prevalent form of find is **breadth-first** search.

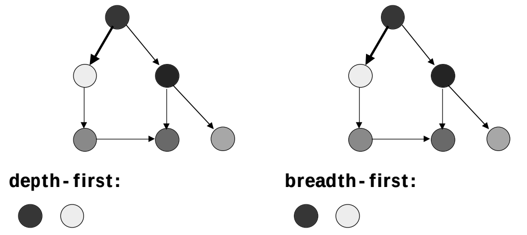
Search Orders



depth-first:

breadth-first:

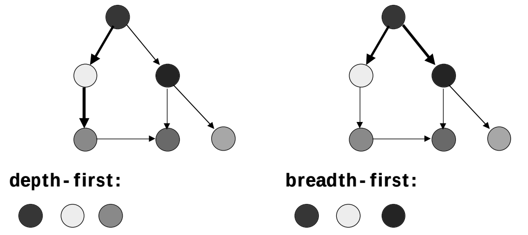
Search Orders



depth-first:

breadth-first:

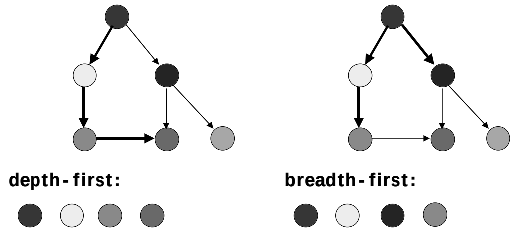
Search Orders



depth-first:

breadth-first:

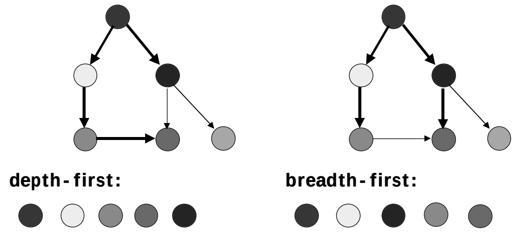
Search Orders



depth-first:

breadth-first:

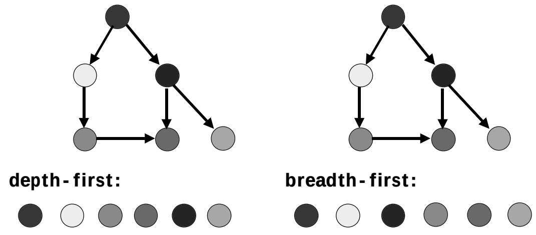
Search Orders



depth-first:

breadth-first:

Search Orders



depth-first:

breadth-first:

Comparative Strengths

- Breadth-first has the advantage of finding the ***shortest*** path to a desired node.
- Depth-first is easier to implement and is more space-efficient.

Underlying Data Structures

- Depth first uses recursion; the underlying structure is a “stack”
- Breadth-first uses a “queue” (first-in-first-out list)
- More on this when we discuss **implementation** of information structures
- See also the discussion in text, sec. 4.15.

Infinite Lists

- In rex, lists can be conceptually infinite.
- Infinite lists allow us to model “processes” that continue processing forever; a list is used as a data stream from one process to another.
- The lists don’t really take up infinite space; they are computed incrementally as needed.

Built-in infinite list functions

- from(0) ➪ [0, 1, 2, 3, 4, ...]
- map(sq, from(0)) ➪ [0, 1, 4, 9, 16, ...]
- map(*, from(0), from(0)) ➪ [0, 1, 4, 9, ...]
- primes() ➪ [2, 3, 5, 7, 11, ...]

Not all functions make sense on infinite lists

- append(from(0), range(1, 10)) ➪ ??

  (be ready with control-C)
- append(range(1, 10), from(11)) is ok
- reverse(from(0)) ➪ ??
  (uh-oh)
- range(0, Infinity) should work

How it’s done: smoke & mirrors

- Delay operator: $
- $ ...expression... means

  “defer computing the value of expression until needed”.
- Application:

  f(g(X), $h(Y))

  where h(Y) is some really expensive computation that might not be needed.

Infinite lists == $?

- from(N) = [N | $ from(N+1)];

```
from(0) => [0 | $ from(1)]
        => [0 | [1 | $ from(2)]]
        => [0 | [1 | [2 | $ from(3)]]]
        => [0 | [1 | [2 | [3 | $ from(4)]]]]

        == [0, 1, 2, 3, 4, ...]
```

A good challenge

- Define *pairs* so that

```
pairs(from(0), from(0)) ==>
   [ [0, 0],
     [0, 1], [1, 0],
     [0, 2], [1, 1], [2, 0], ... ]
```

Every pair must be included eventually.

- A “proof by programming” of a mathematical truth.