Harvey Mudd College
Computer Science 80
Logic for Computer Science
Fall Semester 2001

Assignment #1 – Mathematical Preliminaries
Due 11:00am, Friday September 21, 2001

1. Using the directed graph representation used in class, provide examples of relations on a set of four objects that are:
   (a) transitive, not reflexive, not symmetric
   (b) transitive, symmetric, not reflexive
   (c) symmetric, reflexive, not transitive
   (d) anti-transitive, anti-symmetric, anti-reflexive

2. Define a *non-trivial chain* in a binary relation R as a sequence $(a_1, \ldots, a_n)$ for some $n \geq 2$ such that the a_i are distinct, and $(a_i, a_{i+1}) \in R$ for $i \in [n-1]$. A chain is a *non-trivial cycle* if (a_n, a_1) is also an element of R.

   Prove that a relation is a partial order iff it is reflexive and transitive and has no non-trivial cycles.

3. Given two binary relations, R between A and B, and S between B and C, their *composition*, denoted by $R \circ S$, is a relation between A and C defined by the set of ordered pairs $\{(a, c) | \text{there is a } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$.
   (a) Let $R = \{(a, b), (a, c), (c, d), (a, a), (b, a)\}$. What is the value of $R \circ R$?
   (b) Prove that, given any relation R on a set A, R is transitive iff $R \circ R$ is a subset of R.

4. Given a poset $(A, \leq)$, define the *lexicographic ordering*, $\ll$, on $A \times A$, induced by $\leq$, as follows:

   For all $x, y, x', y' \in A$, $(x, y) \ll (x', y')$ iff either:
   - $x = x'$ and $y = y'$, or
   - $x < x'$, or
   - $x = x'$ and $y < y'$

   Prove that if the poset $(A, \leq)$ is well-founded, then $(A \times A, \ll)$ is also well-founded. (You may assume [though you might want to confirm for yourself] that if $\leq$ is a partial order, then $\ll$ is indeed a partial order).

5. Ackerman's function on $\mathcal{N} \times \mathcal{N}$ is defined recursively in ML as:

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fun A(x,y) = if x = 0
               then y+1
               else if y = 0
                      then A(x-1,1)
                      else A(x-1, A(x,y-1));
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Assuming that such a recursive definition actually defines a partial function (this is non-trivial but is established in *recursive function theory*), prove by induction over the lexicographic ordering of $\mathcal{N} \times \mathcal{N}$ that Ackerman's function is in fact a total function on pairs of natural numbers. That is, that for any pair of inputs there is a defined output.