Decision Procedures

How do we know if a formula $\Phi \in PROP$ is valid?

We need a *decision procedure*. I.e. an algorithm that, given a formula, returns **true** if the formula is valid and **false** if it is not (that is, if it is falsifiable).

Is there such an algorithm that is guaranteed to terminate?

Sure...

Further, Theorem 2.5.11 enables us to extend this procedure to determine if a formula is a logical consequence of a set of formulas.

But there are real problems with this method...

Deduction Systems

A *deduction system* is a syntactic system for manipulating formulas designed to lead to conclusions which correspond to logical consequences.

Formulas are manipulated by way of *deduction rules* intended to mimic *sound* inferences. Each rule has *premises* and a *conclusion*. There are also *axioms* which behave like rules with no premises.

A well-formed deductive structure is called a *proof*, and has some *conclusion*.

A formula, Φ, which may occur as the conclusion of a proof is said to be *provable* (or *derivable*), written $\vdash \Phi$. A proof may contain some formulas which are used as *assumptions*. If there is a proof of Φ using assumptions from the set Γ, we write $\Gamma \vdash \Phi$ (Φ is *derivable from* Γ).

Note: If we are discussing more than one deduction system, then for clarity, if Φ can be derived from Γ using the rules of the system $\mathcal{D}$ (Φ is *derivable from* Γ *under* $\mathcal{D}$), we will write $\Gamma \vdash_{\mathcal{D}} \Phi$.

Soundness and Completeness

The goal is to create deduction systems in which $\Gamma \vdash \Phi$ iff $\Gamma \models \Phi$.

Definition: Given a deduction system, $\mathcal{D}$, if $\Gamma \vdash_{\mathcal{D}} \Phi$ implies $\Gamma \models \Phi$, we say that $\mathcal{D}$ is *sound.*

Definition: Given a deduction system, $\mathcal{D}$, if $\Gamma \models \Phi$ implies $\Gamma \vdash_{\mathcal{D}} \Phi$, we say that $\mathcal{D}$ is *complete*.

Natural Deduction

In 1935, Gerhard Gentzen, proposes a new way of formulating logic.

In *Investigations Into Logical Deduction* he lays out what amounts to a particular collection of derived rules for what is known as *Axiomatic* or *Hilbert* systems, but then shows that these rules, without any axioms, are themselves complete.

These rules also correspond more closely to a natural way of writing proofs, and he therefore calls the new system *Natural Deduction*.

Natural Deduction

There are two rules for each operator, an *Introduction Rule*, and an *Elimination Rule*.

Introduction Rules

$$\frac{\begin{matrix}\not\psi \\ \vdots \\ \phi\end{matrix}}{\psi \Rightarrow \phi} \Rightarrow_I$$

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge_I$$

$$\frac{\phi}{\phi \vee \psi} \vee_{I_1} \qquad \frac{\psi}{\phi \vee \psi} \vee_{I_2}$$

$$\frac{\begin{matrix}\not\psi & \not\phi \\ \vdots & \vdots \\ \phi & \psi\end{matrix}}{\psi \equiv \phi} \equiv_I$$

$$\frac{\begin{matrix}\not\phi \\ \vdots \\ \bot\end{matrix}}{\neg\phi} \neg_I$$

Elimination Rules

$$\frac{\psi \quad \psi \Rightarrow \phi}{\phi} \Rightarrow_E$$

$$\frac{\phi \wedge \psi}{\phi} \wedge_{E_1} \qquad \frac{\phi \wedge \psi}{\psi} \wedge_{E_2}$$

$$\frac{\begin{matrix} & \not\phi & \not\psi \\ & \vdots & \vdots \\ \phi \vee \psi & \xi & \xi\end{matrix}}{\xi} \vee_E$$

$$\frac{\psi \quad \psi \equiv \phi}{\phi} \equiv_E \qquad \frac{\phi \quad \psi \equiv \phi}{\psi} \equiv_E$$

$$\frac{\phi \quad \neg\phi}{\bot} \neg_E$$

$$\frac{\bot}{\phi} \bot_E$$

Some Natural Deduction Proofs

$$\frac{\vdots}{p \Rightarrow p}$$

$$\frac{\vdots}{(a \wedge b) \Rightarrow (b \wedge a)}$$

$$\frac{\vdots}{(a \vee b) \Rightarrow (b \vee a)}$$

Some Natural Deduction Proofs

$$\frac{\vdots}{((hj \Rightarrow (bij \Rightarrow bd)) \Rightarrow (hj \Rightarrow (bij \Rightarrow bd)))}$$

$$\frac{\vdots}{(hj \Rightarrow (bij \Rightarrow ((hj \Rightarrow (bij \Rightarrow bd)) \Rightarrow bd)))}$$

Some Natural Deduction Proofs

$$\Xi_1 = \frac{\vdots}{((\phi \vee \sigma) \wedge (\psi \vee \sigma))}$$

$$\Xi_2 = \frac{\vdots}{((\phi \wedge \psi) \vee \sigma)}$$

$$\frac{\begin{array}{cc} \Xi_1 & \Xi_2 \\ ((\phi \vee \sigma) \wedge (\psi \vee \sigma)) & ((\phi \wedge \psi) \vee \sigma) \end{array}}{((\phi \wedge \psi) \vee \sigma) \equiv ((\phi \vee \sigma) \wedge (\psi \vee \sigma))} \equiv_R$$