# Gentzen Sequent Calculus

A **Sequent** is a syntactic structure of the form:

$$\Gamma \longrightarrow \Delta$$

where $\Gamma, \Delta$ are (possibly empty) comma-delimited sets of well-formed formulas.

You can read an individual sequent as a meta-statement about a logical consequence:

> If all of the formulas in $\Gamma$ are true, then at least one of the forumlas in $\Delta$ is true.

Note that this is not the same as saying:

> At least one of the formulas in $\Delta$ is a logical consequence of $\Gamma$.

Why not?

We will begin by looking at the rules for the intuitionistic calculus, in which the right-hand-side, $\Delta$ is restricted to being at most one formula.

# LJ – Intuitionistic Gentzen Sequent Calculus

Assume $\Gamma, \Delta$ are (possibly empty) sets of wffs, and $A, B, C$ are wffs.

The rules can be read from the top down (giving instructions on how to construct a sound deduction) or bottom up (giving instructions on how to search for a proof):

$$\frac{}{\Gamma, A \longrightarrow A}\ id \qquad \frac{}{\Gamma \longrightarrow \top}\ \top \qquad \frac{}{\Gamma, \bot \longrightarrow A}\ \bot$$

$$\frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge_L \qquad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge_R$$

$$\frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee_L \qquad \frac{\Gamma \longrightarrow A_i}{\Gamma \longrightarrow A_1 \vee A_2}\ \vee_R$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \Rightarrow B \longrightarrow C}\ \Rightarrow_L \qquad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \Rightarrow B}\ \Rightarrow_R$$

$$\frac{\Gamma \longrightarrow A}{\Gamma, \neg A \longrightarrow C}\ \neg_L \qquad \frac{\Gamma, A \longrightarrow \bot}{\Gamma \longrightarrow \neg A}\ \neg_R$$

# LK – Classical Gentzen Sequent Calculus

To reason classically, we simply expand the right-hand-side of the sequent to a set of formulas:

$$\frac{}{\Gamma, A \longrightarrow \Delta, A}\,id \qquad \frac{}{\Gamma \longrightarrow \Delta, \top}\,\top \qquad \frac{}{\Gamma, \bot \longrightarrow \Delta}\,\bot$$

$$\frac{\Gamma, A, B \longrightarrow \Delta}{\Gamma, A \wedge B \longrightarrow \Delta}\,\wedge_L \qquad \frac{\Gamma \longrightarrow \Delta, A \quad \Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, A \wedge B}\,\wedge_R$$

$$\frac{\Gamma, A \longrightarrow \Delta \quad \Gamma, B \longrightarrow \Delta}{\Gamma, A \vee B \longrightarrow \Delta}\,\vee_L \qquad \frac{\Gamma \longrightarrow \Delta, A, B}{\Gamma \longrightarrow \Delta, A \vee B}\,\vee_R$$

$$\frac{\Gamma \longrightarrow \Delta, A \quad \Gamma, B \longrightarrow \Delta}{\Gamma, A \Rightarrow B \longrightarrow \Delta}\,\Rightarrow_L \qquad \frac{\Gamma, A \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, A \Rightarrow B}\,\Rightarrow_R$$

$$\frac{\Gamma \longrightarrow \Delta, A}{\Gamma, \neg A \longrightarrow \Delta}\,\neg_L \qquad \frac{\Gamma, A \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \neg A}\,\neg_R$$

$$\frac{\Gamma \longrightarrow \Delta, A, B \quad \Gamma, A, B \longrightarrow \Delta}{\Gamma, A \equiv B \longrightarrow \Delta}\,\equiv_L \qquad \frac{\Gamma, A \longrightarrow \Delta, B \quad \Gamma, B \longrightarrow \Delta, A}{\Gamma \longrightarrow \Delta, A \equiv B}\,\equiv_R$$

# Some Sequent Calculus Proofs

$$\frac{\vdots}{\longrightarrow p \Rightarrow p}$$

$$\frac{\vdots}{\longrightarrow (a \wedge b) \equiv (b \wedge a)}$$

# Some Sequent Calculus Proofs

$$\frac{\vdots}{\longrightarrow (a \vee b) \equiv (b \vee a)}$$

# Some Sequent Calculus Proofs

$$\frac{\vdots}{\longrightarrow (hj \Rightarrow (bij \Rightarrow ((hj \Rightarrow (bij \Rightarrow bd)) \Rightarrow bd)))}$$

# Some Sequent Calculus Proofs

$$\Xi_1 = \frac{\vdots}{((\phi \wedge \psi) \vee \sigma) \longrightarrow ((\phi \vee \sigma) \wedge (\psi \vee \sigma))}$$

$$\Xi_2 = \frac{\vdots}{((\phi \vee \sigma) \wedge (\psi \vee \sigma)) \longrightarrow ((\phi \wedge \psi) \vee \sigma)}$$

$$\frac{\begin{array}{cc} \Xi_1 & \Xi_2 \\ ((\phi \wedge \psi) \vee \sigma) \longrightarrow ((\phi \vee \sigma) \wedge (\psi \vee \sigma)) & ((\phi \vee \sigma) \wedge (\psi \vee \sigma)) \longrightarrow ((\phi \wedge \psi) \vee \sigma) \end{array}}{\longrightarrow ((\phi \wedge \psi) \vee \sigma) \equiv ((\phi \vee \sigma) \wedge (\psi \vee \sigma))} \equiv_R$$

# Some Sequent Calculus Proofs

$$\dfrac{\vdots}{\longrightarrow (a \Rightarrow (b \wedge c)) \Rightarrow (a \Rightarrow b)}$$