Logic and Mathematics

To apply FOPC to mathematical reasoning, we must *axiomatize* the structures about which we wish to reason. That is, we write down a set of first-order sentences (formulas with no free variables), Γ, that describe the structure. Any model of Γ is then such a structure.

To deduce properties of a structure, we ask what formulas are logical consequences of Γ, i.e., for what A does $\Gamma \models A$.

Today we will just look at examples of writing down such axiomatizations.

One question of interest to logicians is whether a system has a *finite axiomatization*, or not.

It is also interesting to discover that there are mathematical properties of structures that can be determined just by looking at the meta-structure of their axiomatization.

Partial Orders

Definition: A structure is a *partially ordered set* if it is a set together with a reflexive, transitive, anti-symmetric relation.

Language:

$\mathcal{A} = \{ \qquad\qquad \} \quad \mathcal{F} = \{ \qquad\qquad \} \quad \mathcal{P} = \{ \qquad\qquad \}$

Γ**:**

Definition: A poset is *totally* or *linearly ordered* if all elements are comparable.

Γ**:**

Definition: A poset is *densely ordered* if between any two distinct elements there is a third one.

Γ**:**

Partial Orders

Definition: A poset has an *upper bound* if there is an element at least as big as any other element.

Γ**:**

Definition: A poset has a *least upper bound* if it has an upper bound that is as small as any upper bound.

Γ**:**

Note that any property that requires refering to subsets as individuals is second-order, meaning it is not axiomatizable as a set of first-order sentences.

Peano Arithmetic

Peano laid out the following set of axioms, intending them to describe the natural numbers under successor, addition, and multiplication (up to isomorphism). It turns out, however, that they are modeled by a large class of non-isomorphic structures (called arithmetic structures or Peano structures).

Language:

$\mathcal{A} = \{ \qquad \}$ $\quad \mathcal{F} = \{ \qquad \}$ $\quad \mathcal{P} = \{ \qquad \}$

Γ**:**

- Zero is the successor of no number.

- If the successors of two numbers are equal, then the two numbers are equal.

- The sum of any number and zero is that number.

Peano Arithmetic

- The sum of a number and the successor of another number is the successor of the sum of the two numbers.

- The product of any number and zero is zero.

- The product of any number and the successor of another number is the sum of the first number and the product of the two numbers.

Peano Arithmetic

- (The Principle of Mathematical Induction)
  If a property holds for zero, and holds for the successor of a number whenever it holds for the number, then it holds for all numbers.

Peano Arithmetic

We can write down a variety of properties of numbers as the definitions of predicates. These definitions can be at the meta-level, as a sort of macro, or at the object level as new axioms.

- A number is even if:

- A number is divisible by another number if:

- A number is prime if: