Towards First-Order Resolution

We'll now start the development of First-Order Resolution.

As with everything else we've looked at in first-order, this will be structurally similar to the propositional case, but with some significant complications. The development will take two or three classes.

The key to making resolution work will be the same as the trick we noticed in the presentation of the Gentzen Sequent Calculus: We need some way to delay the choice of terms we use to instantiate the quantified variables.

That technique was the breakthrough described by J.A. Robinson in 1963. However, it depends on theoretical foundations established by Gödel, Lowenheim, Skolem, and Herbrand over the prior thirty years.

Prenex Conjunctive Normal Form

As in propositional resolution, we will simplify the proof-search algorithm by reducing the formulas under consideration to a simplified normal form. This will eliminate the need to consider many different operator rules, and leave us with a single "resolution" rule.

Definition: (3.8.1) A formula is in *prenex conjunctive normal form* (*PCNF*) iff it is of the form:

$$Q_1 x_1 \ldots Q_n x_n (M)$$

where the Q_i are quantifiers and M is a quantifier-free formula in conjunctive normal form. We call the sequence of quantifiers the *prefix*, and the underlying formula M the *matrix*.

Prenex Conjunctive Normal Form

Theorem: Given any formula $\Phi \in FORM$, there is a formula $\Phi' \in FORM$ such that Φ' is in PCNF, and $\Phi \leftrightarrow \Phi'$.

Proof. The proof is in the form of an algorithm which constructs the equivalent PCNF formula.

1. Rename all bound variables so that they are unique.
2. Replace uses of $\Rightarrow$ and $\equiv$ with uses of $\neg$, $\vee$, and $\wedge$.
3. Push negations inwards, using the established dualities:
$$\neg(\Phi \wedge \Psi) \leftrightarrow (\neg\Phi) \vee (\neg\Psi)$$
$$\neg(\Phi \vee \Psi) \leftrightarrow (\neg\Phi) \wedge (\neg\Psi)$$
$$\neg(\forall x(\Phi)) \leftrightarrow \exists x(\neg\Phi)$$
$$\neg(\exists x(\Phi)) \leftrightarrow \forall x(\neg\Phi)$$
4. Move the quantifiers to the outside.
5. Use distributivity equivalences to put matrix in CNF.

Dealing With Quantifiers

To further simplify the representation and treatment of the quantifiers it would be really nice if it were possible to only have one kind of quantifier. Then we could drop the explicit quantifiers and assume that all variables are bound by that sort of quantifier in the prefix. It would also simplify resolution since there would only be one quantifier rule whose logic would need to be encoded in the resolution rule.

Unfortunately, it is not as simple as just replacing the unwanted quantifier with its DeMorgan dual (i.e. $\exists x(\Phi) \leftrightarrow \neg\forall x(\neg\Phi)$).

Why not?

Fortunately, Skolem's theorem provides a solution.

Skolem's Theorem

Definition: (3.8.2) A formula is in *clausal form* if it is in PCNF and the prefix consists only of universal quantifiers.

Theorem (3.8.3 [Skolem's Theorem])**:** Given any *closed* formula $\Phi \in FORM$, there is a formula $\Phi' \in FORM$ sucht that Φ' is in clausal form, and $\Phi \approx \Phi'$ (that is, Φ is satisfiable iff Φ' is satisfiable.).

Note that the theorem does not state that there is a logically equivalent clausal formula (as there is for PCNF), only that there is one which shares Φ's satisfiability (which we use to establish the validity of $\neg\Phi$).